Playing with Polygons: Interactive Explorations of Mathematical and Artistic Patterns

A book proposal by Stephen Erfle, Dickinson College

This is a book that encourages independent mathematical exploration using regular polygons as the platform for analysis. These explorations are based on a series of Excel workbooks created to examine various aspects of regular polygons. Each file was created in order to focus the user's attention on the topic at hand, rather than the nuts and bolts of creating the various images in the file. This allows quicker and more fluid exploration of ideas than is possible if creating images by hand. Therefore, pattern recognition and mathematical understanding deepens.

Target audience. There are multiple audiences for this book. One could consider this a book of recreational mathematics, not mathematical research or a mathematics textbook. It is certainly useful for those interested in mathematics education. But it would also be helpful in bridging the gap between mathematics and art. Although mathematics and art teachers in K-12 could certainly incorporate these materials into their classroom, they were initially developed for independent exploration outside the classroom. The idea was to create materials that were easy enough to use that students would want to explore each model and create their own patterns by manipulating the parameters of each model.

Why Excel and how much familiarity with Excel do readers need to have? Excel is a powerful spreadsheet program that is very widely available due to the prominence of Microsoft Office suite. The files require Excel to operate, but this does not mean that readers need to know how to use Excel to use the files. Most require nothing more than entering a number in a cell, scrolling values up or down with a scrollbar, or clicking on or off a click-box to manipulate the image shown. These simple manipulations do not require Excel expertise, and because they are so easy to use, they allow readers to focus on the patterns that result.

Why regular polygons? Regular polygons are among the earliest geometric figures that students are exposed to and they form some of the building blocks used for a number of topics later in their mathematical education. They are ubiquitous.

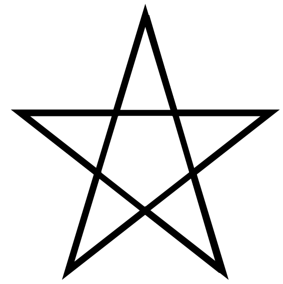
This book pulls together topics at various levels of mathematical sophistication. Given the highly visual nature of the material, students are able to enjoy exploring a variety of topics even when they do not fully understand the underlying mathematics. Such explorations are attuned with Sherman Stein's suggestion that mathematical understanding is best accomplished by following what he called *The Triex: Explore, Extract, Explain* (Stein, 1996). Stein argues that we should encourage our students to explore and gather data, seek to extract some sort of order or pattern from those explorations, and ultimately seek an explanation for what has been found. He points out that even if students are unable to attain that third step, they have been primed to appreciate the explanation for why the pattern exists provided by the instructor or another student (Stein, 1996, p. 6). Such explorations encourage learning and that learning can occur outside the classroom even in the absence of complete understanding of the results.

Layout of the book. The book will have an introductory portion of one to two chapters followed by two major parts. The first part is based on string art, sometimes called aestheometry. This part is more highly artistic. The second part is based on parallel lines that one can create using the vertices of the polygon. This part is more strongly mathematical in the sense that it will focus attention on counting rules (see file 7 below for a clear example of what this means).

Each chapter will highlight topics and questions that are best examined via a specific file, but the files build off one another so that parameters learned using one file will help with later files. A preliminary list of these files and the topics one can explore with each file is listed because many files can be used for more than one chapter. Sample images produced and examples of explorable topics follow. Each chapter will include challenge questions that ask the reader to replicate images using the file. This will deepen understanding of the underlying parameters of the model. Examples of challenge questions are shown at the end of the introduction to aestheometry file discussion (file 2).

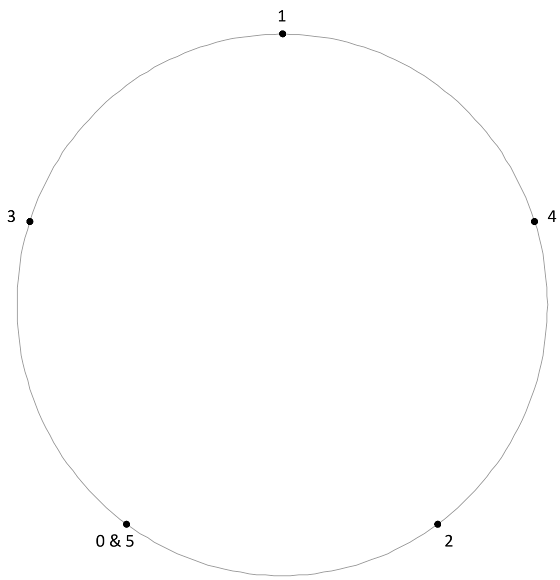
The introductory portion lays out polygon and star basics. We will focus on continuously drawn stars meaning that you create the image by drawing from vertex to vertex without lifting your pencil until the initial vertex is once again used. Once that occurs, the circuit is complete.

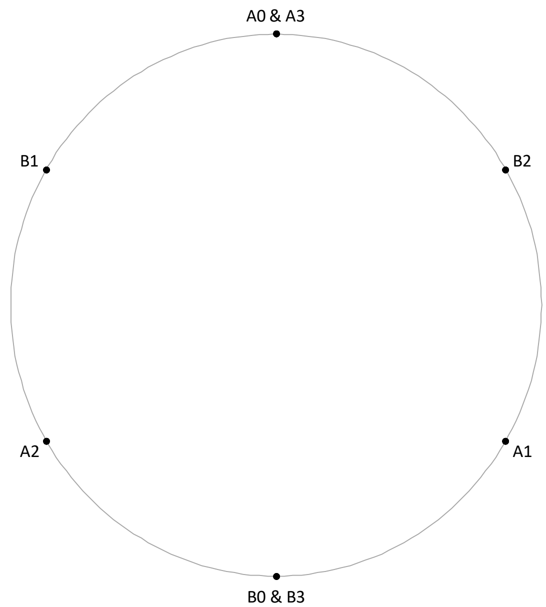
The simplest example is a *pentagram*, or five-point star. A casual survey of more than 100 college students (which included a number of international students as well as students from various parts of the country) were asked without further prompting to draw a star. Then they were asked to think about where they started (Top, Upper Left, Upper Right, Lower Left, Lower Right) and whether they drew the star in a clockwise or counter-clockwise fashion. There are ten possible answers (5 points $\times$ 2 directions).



The results were surprising: more than 80% of students sampled drew a star *exactly* the same way. They started at the lower left then moved clockwise around the circle, jumping over the next vertex (upper left) to draw the first line to the top. The pattern followed is shown on the left where you simply have to follow the numbers from 0 to 5 to draw the star. (Most of the rest started at the top and less than 5% traced out the image in a counterclockwise fashion.)

The only *n*-sided polygon with $n > 4$ where one cannot create a continuously drawn star using all vertices is $n = 6$, a hexagon. The problem is shown on the right below. One can draw a 6-point star but only by completing two separate circuits, *A* and *B*. (By contrast, there are two continuously drawn 7-point stars but only one 8-point star.) This and other topics are examined in the introductory part of the book.





Organization of files in the text

1. Polygons Basics (based on EWEP, 2021, "Connecting Geometric Patterns to Numeric Patterns using the *Polygons and Stars* Excel File," *Spreadsheets in Education* targeted K-2)

PART I. Aestheometry (electronic string art) on Polygons and Stars

2. Introduction to Aestheometry on Polygons (single vertex jump) and Stars (multiple jumps)
3. Center-pointed flowers
4. Two Jump Patterns
5. Four color model with 1 to 3 jump patterns on dodecagon (an extension of E&E *Bridges 2020* article)
6. Spirals and more (based on workshop paper under review to *Bridges* 2021, targeted 3-5)

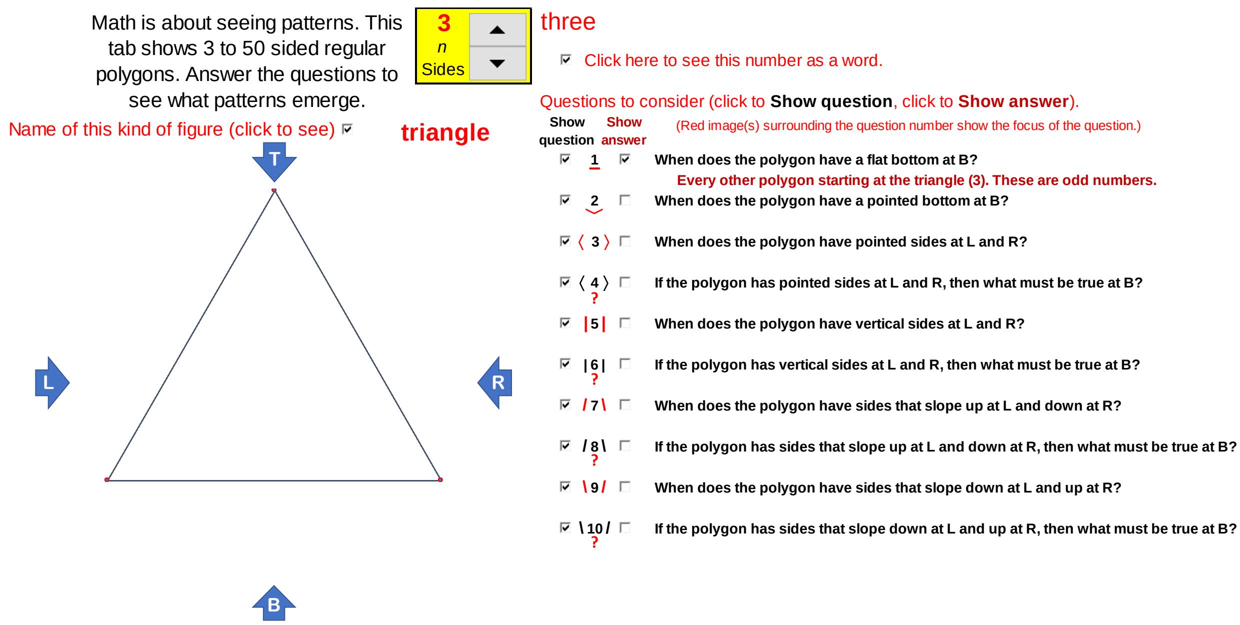
PART II. Using polygonal vertices to frame parallel lines

7. Triangles and perfect squares (based on E&C *Alternative Visions of Perfect Squares*, targeted 3-5)
8. General triangular model
9. Counting sharpest triangles

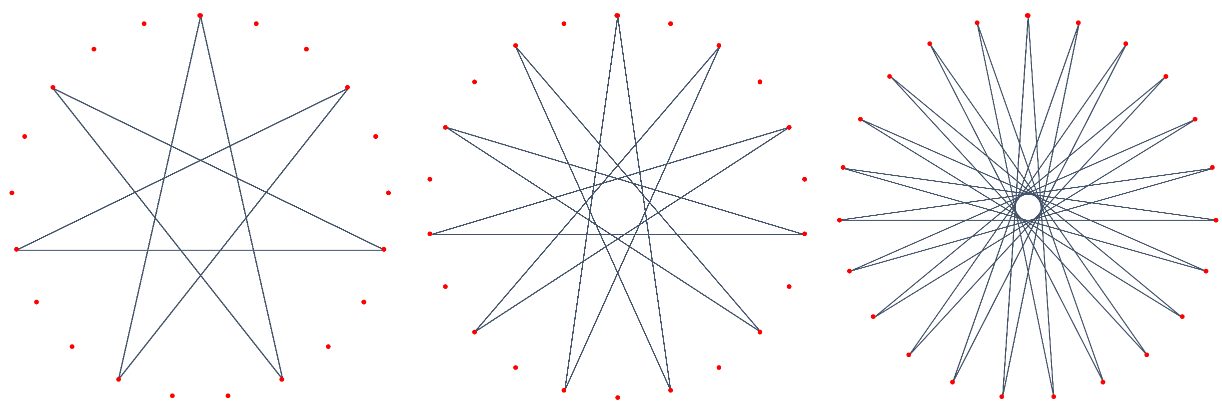
1. Polygon basics: Based on Erfle, Wensel, Erfle and Polinka "Connecting Geometric Patterns to Numeric Patterns using the *Polygons and Stars* Excel File," *Spreadsheets in Education*, 2021, https://sie.scholasticahq.com/article/21267-teaching-number-patterns-with-polygons-and-stars

Abstract: This paper describes a novel approach for exploring counting and pattern recognition in early elementary classrooms that ties geometric patterns to numeric patterns. Active-learning exercises that start out face-to-face (or using the virtual *Class Circle* sheet of the *Polygons and Stars* Excel file) can be explored quickly and more fully using the *Polygons* and *Stars* sheets. This file can be used by the teacher in the classroom, as well as by students undertaking independent explorations. Such explorations help energize young learners; as they explore, extract, and explain what they have found, they begin to recognize the beauty inherent in mathematical patterns.

Polygons sheet showing 10 hideable questions. The parameter *n* is the number of sides to the polygon.



The *Stars* sheet shows what happens when you jump over vertices. Below are 3 examples of stars with *J* being the number of jumps (counted clockwise) between successive lines (*J* = 12 for *n* = 21, 22, and 23)

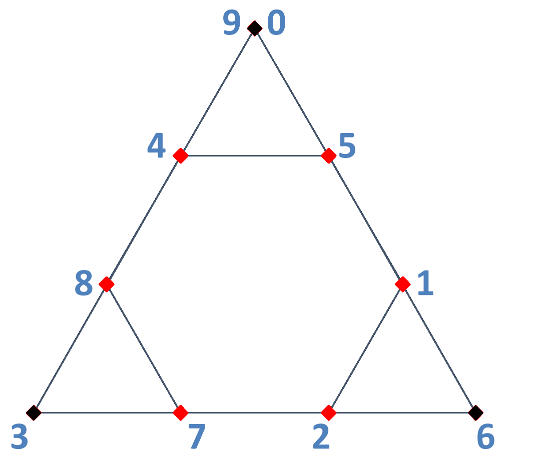


The *Stars* sheet allows users to make continuously drawn stars with from 5 to 50 points. All polygonal vertices are used when the greatest common divisor (GCD) between *n* and *J* (image on right) is 1, but when a common divisor exists, the number of points to the star is *n*/GCD(*n*, *J*). GCD(*n*, *J*) is 3, 2, and 1 above so that the stars have 7 = 21/3, 11 = 22/2, and 23 = 23/1 points respectively. K-2 students do not know about division, but they can see that only 1/3 and ½ of the vertices are used in the first two images and they can count points on the resulting star. As noted above, a continuously drawn star can be created for all *n* > 4 except *n* = 6. To create a 6-point star one needs two equilateral triangles rotated 180° from one another.

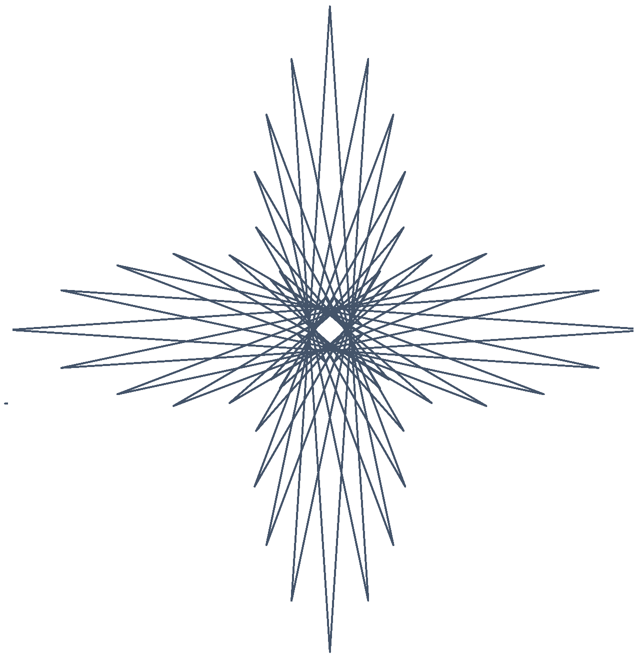
2. Introduction to Aestheometry on Polygons and Stars: An extension of Polygons and Stars introduces the distinction between *subdivisions between vertices*, *S*, and *subdivisions between points*, *P*. There are 3 subdivisions (*S* = 3) for the two triangular (*n* = 3) frames on the left below, the difference is *P* = 2 for the top image and *P* = 4 below. Follow the numbers from 0 to 9 in both to see how both patterns emerge.

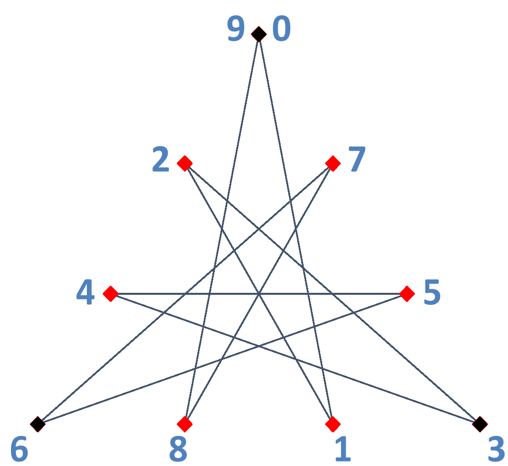
Use the up and down arrows to create your own design like the one below (***S* = 13**, ***P* = 12**, ***J* = 23** and ***n* = 48**)

Follow the numbers **1** to **9** in the figures below to see how ***P*** works using ***S* = 3** and ***n* = 3**.

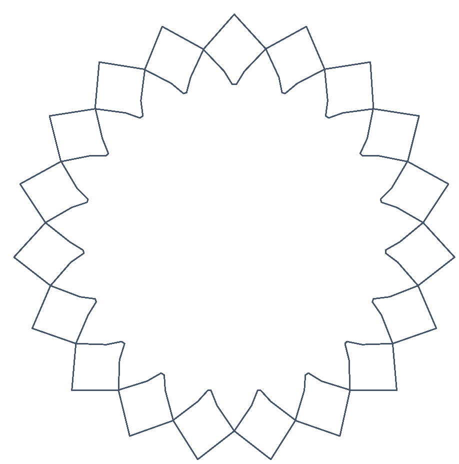


This is ***P* = 2** just like on CurvedTipStars1





This is *P* = 4



7 ***S*, # of segments between vertices**	**27** ***P*, # of segments between Points**	**10** ***J* vertex jumps between points**	**Here you control *P* and *S* independently**	**38** ***n* possible vertices**

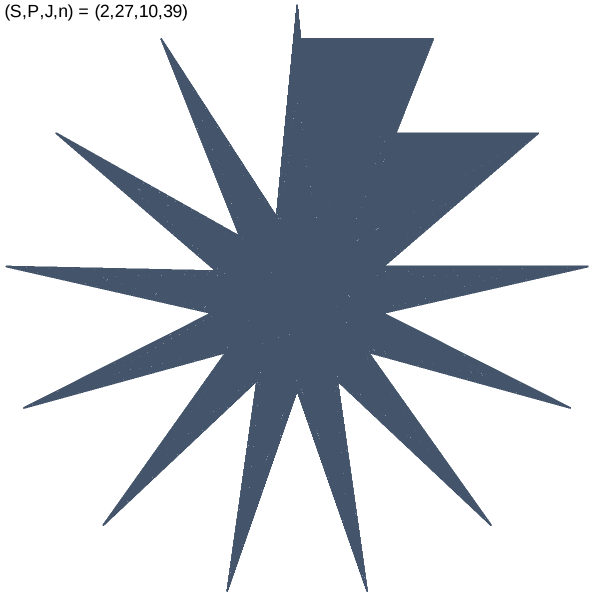
Number of parts in the image 133

(some parts may be on the same line)

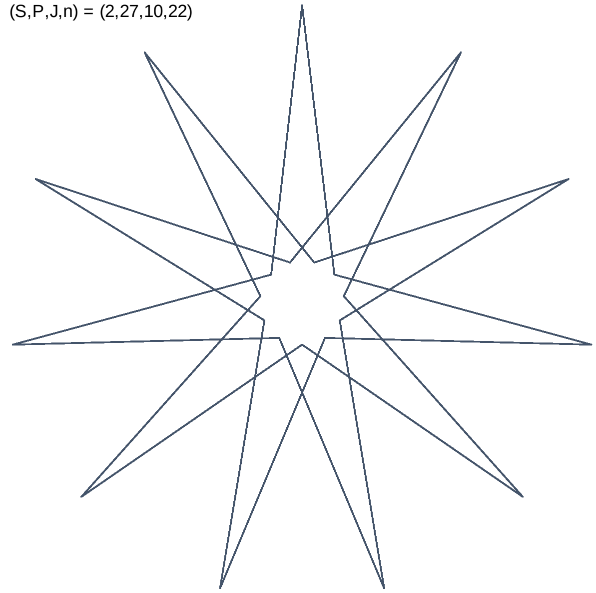
***NOTE:* If the image looks incomplete, move S or P up or down**

These are the dashboard values that produced the lower right image which has 19 petals with 7 lines on each petal (GCD(38,10) = 2, 19 = 38/2, and (7*19 = 133). Change *n* from 3 to 50 and see what happens. The first aestheometry sheet introduces *S* (*P*(*S*) is provided in that sheet); the second aestheometry sheet allows *S* and *P* to be independently controlled. The images on the next page are simply examples showing interesting outcomes. For each, it is instructive to adjust each parameter and see what happens. Notice the star inside a star aspect to the first three images.

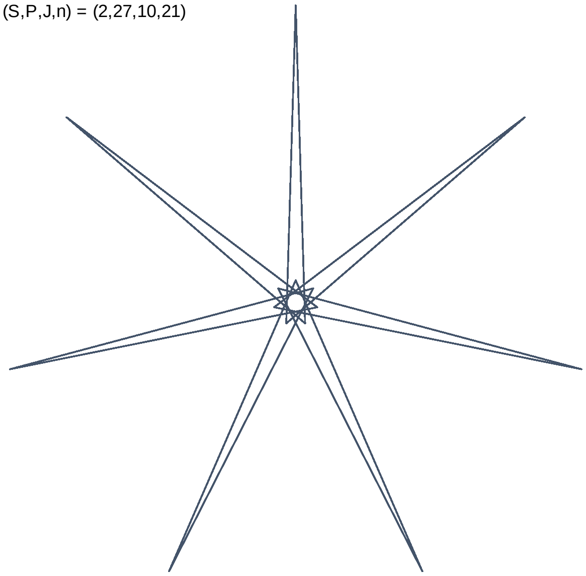
(S,P,J,n) = (2,27,10,39)



(S,P,J,n) = (2,27,10,22)

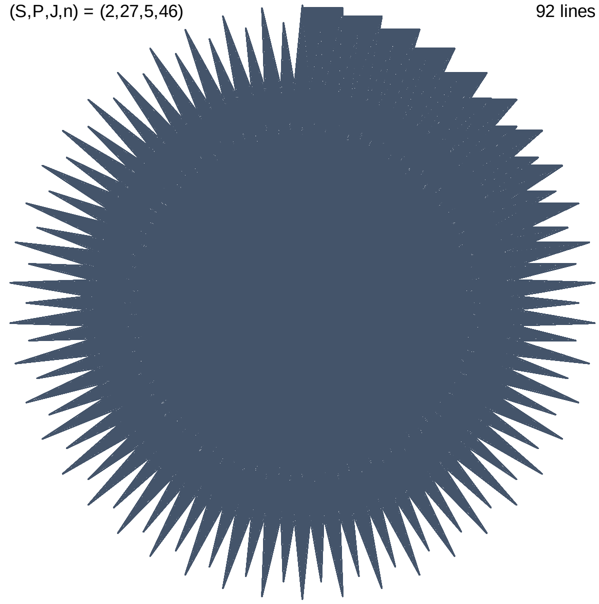


(S,P,J,n) = (2,27,10,21)



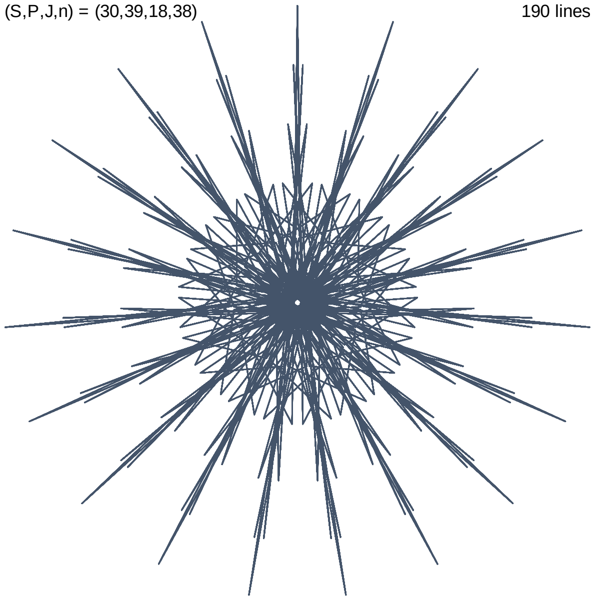
(S,P,J,n) = (2,27,5,46)

92 lines



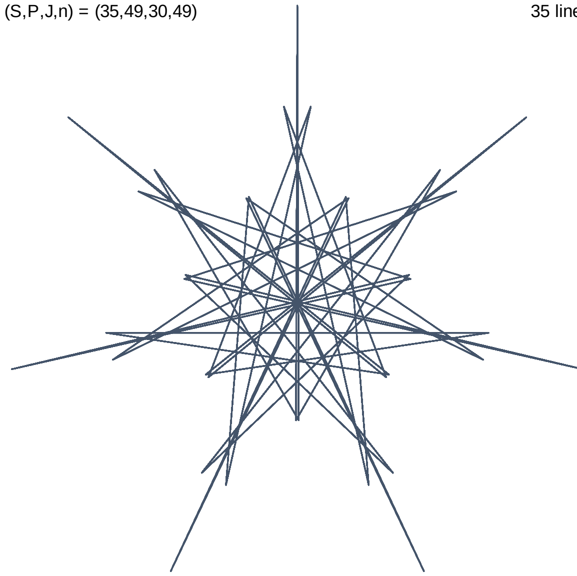
(S,P,J,n) = (30,39,18,38)

190 lines



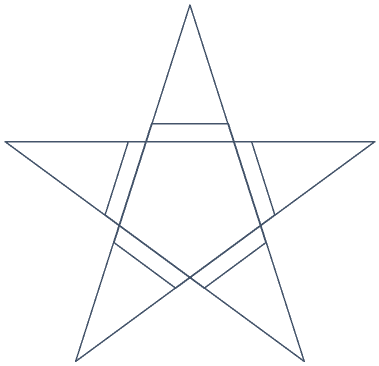
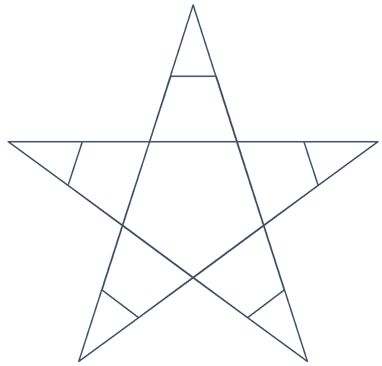
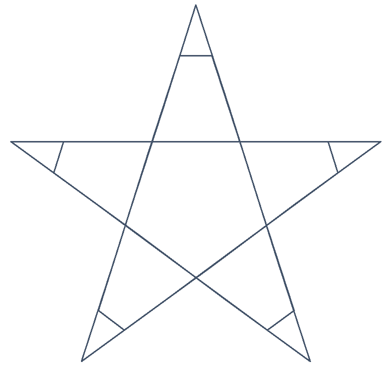
(S,P,J,n) = (35,49,30,49)

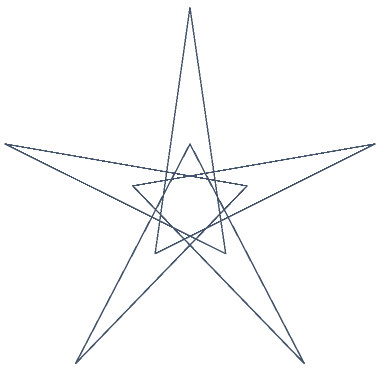
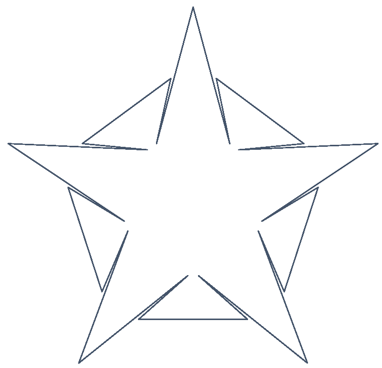
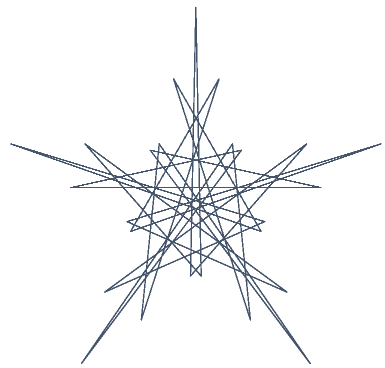
35 lines



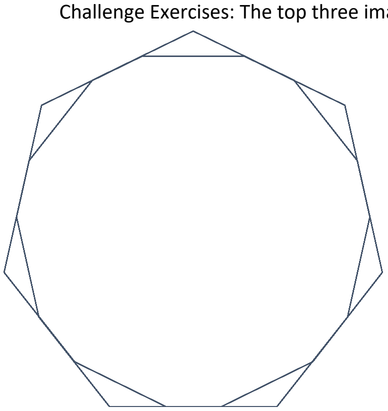
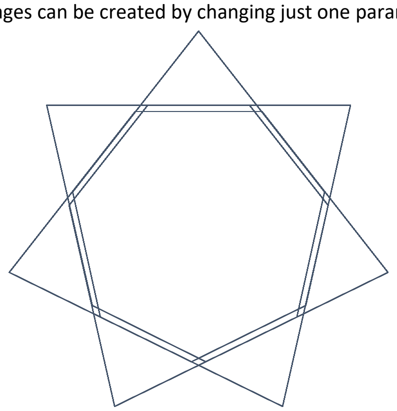
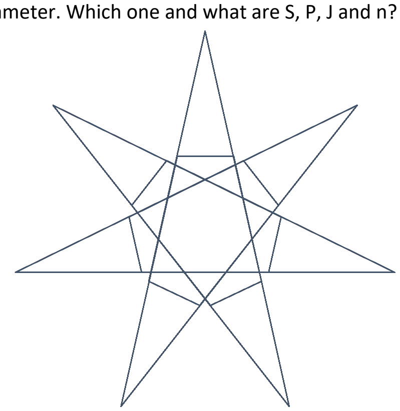
These are best shown in landscape. There are two pages here, compressed into one. Answers below.

K-2 Exercise Using the Curved Tip Star 2 sheet: Find values of S, P, J (vertex jumps) and n (# of vertices) that produce each of these images.

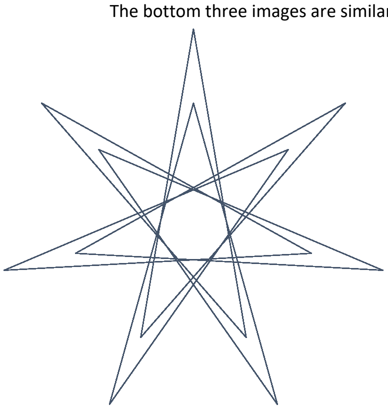
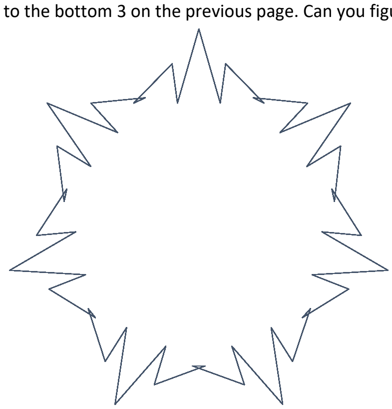
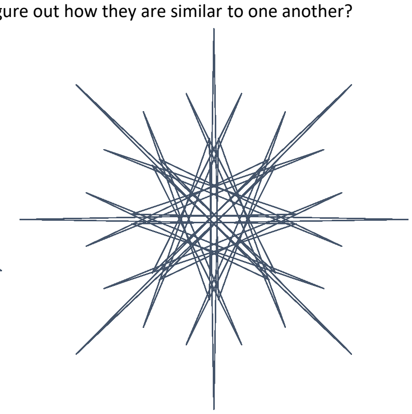




Challenge Exercises: The top three images can be created by changing just one parameter. Which one and what are S, P, J and n?

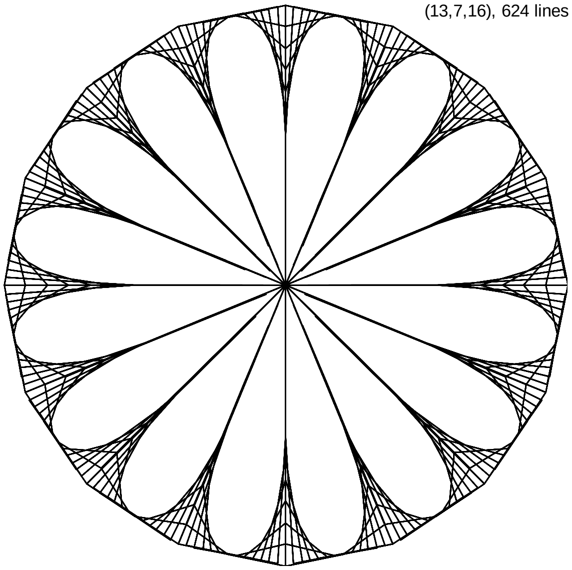




The bottom three images are similar to the bottom 3 on the previous page. Can you figure out how they are similar to one another?

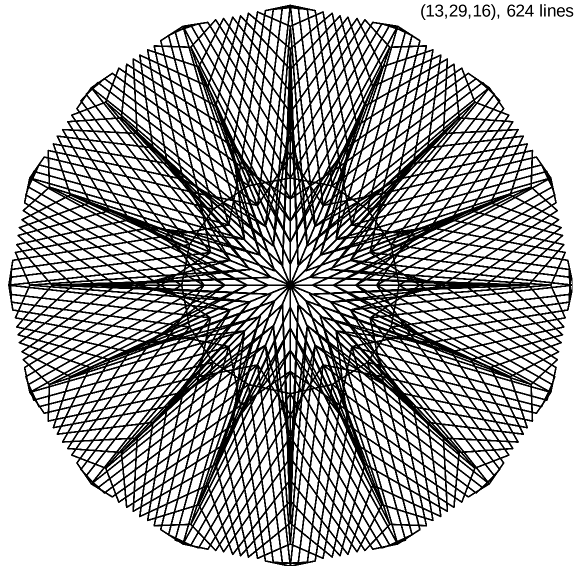




(*S*,*P*,*J*,*n*)	(3,2,2,5)	(5,2,2,5)	(7,2,2,5)
answers	(2,3,2,5)	(5,12,2,5)	(5,7,2,5)
by row:	(3,2,1,7)	(3,2,2,7)	(3,2,3,7)
	(2,3,2,7)	(7,24,2,7)	(5,7,3,8)

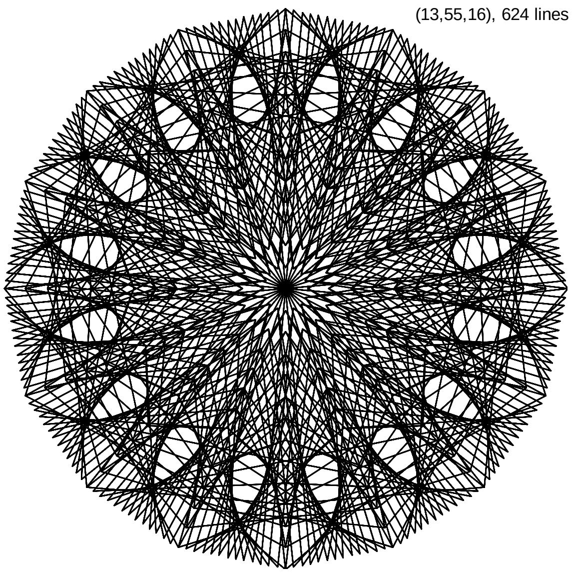
3. Center-pointed flowers. This file forces the center of the circle between successive polygonal vertices. The resulting (*S*, *P*, *n*) image is a flower when *P* < *S.* The images below vary *P* given *S* = 13 and *n* = 16. When *P* = 311, there is a 312 point starburst but when *P* = 312 (half of 624), a single line appears.



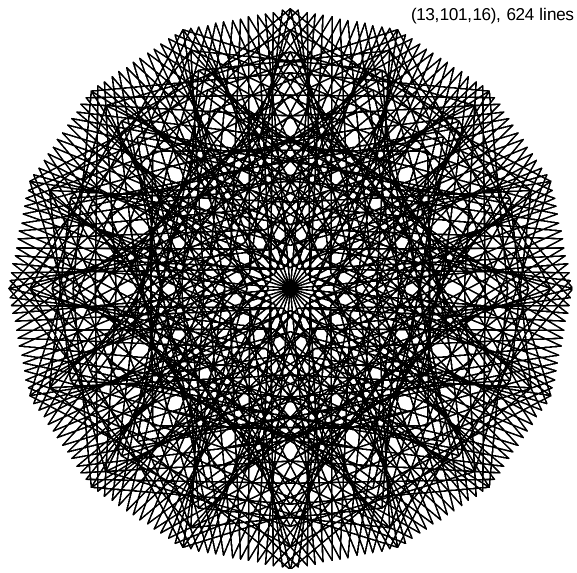
(13,7,16), 624 lines



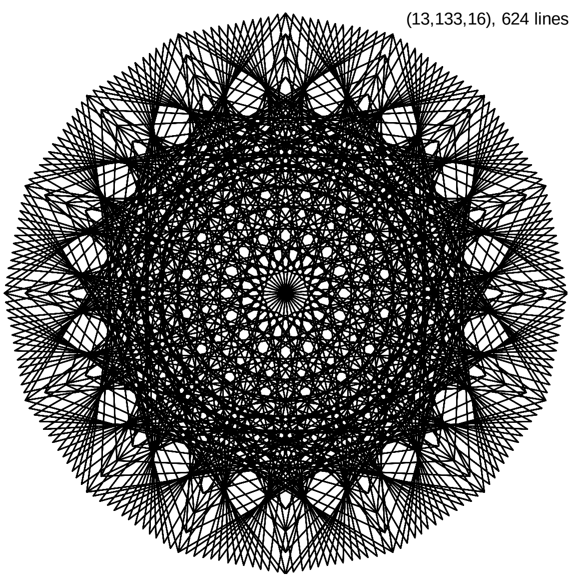
(13,29,16), 624 lines



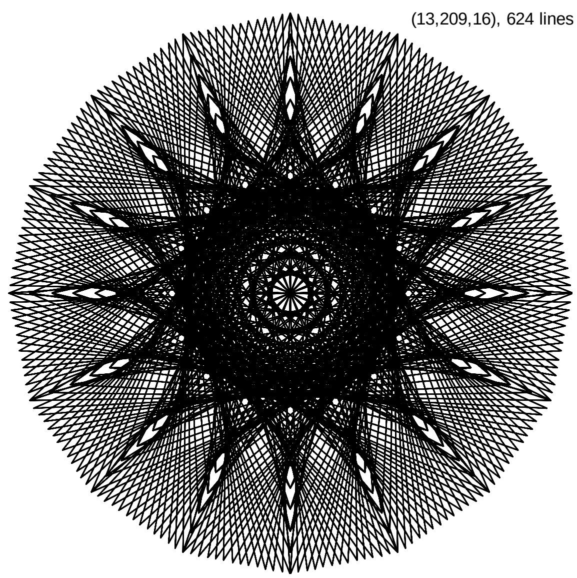
(13,55,16), 624 lines



(13,101,16), 624 lines

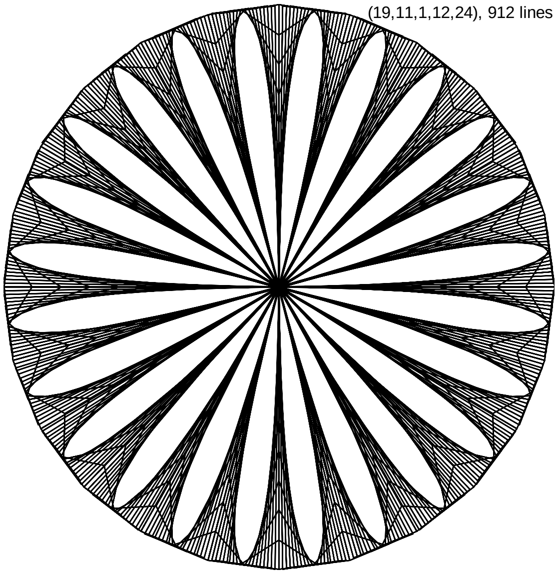


(13,133,16), 624 lines

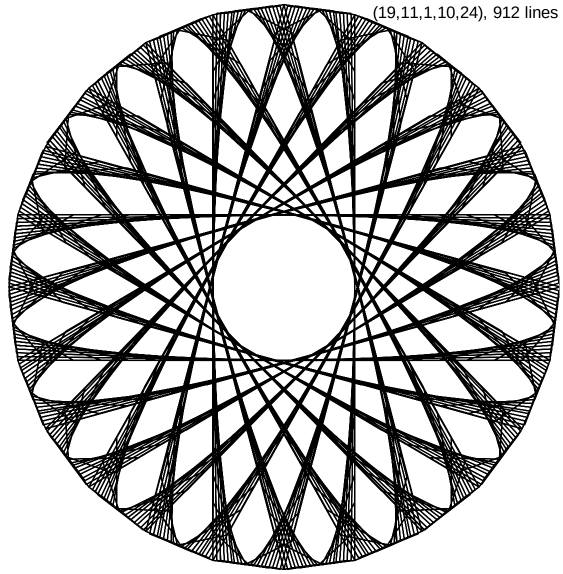


(13,209,16), 624 lines

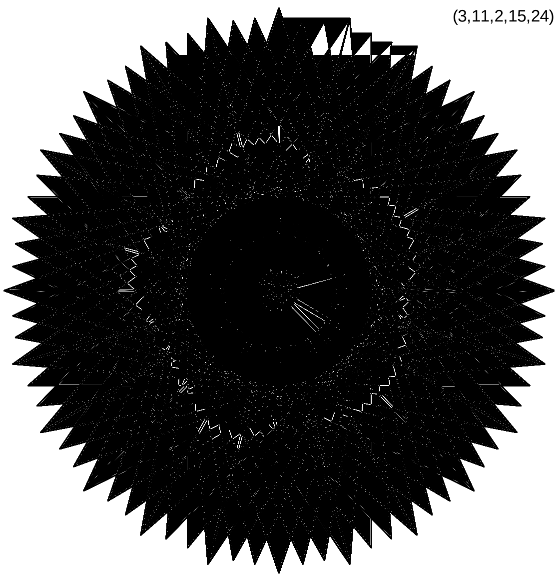
4. Two Jump Patterns. This allows a second level jump, the first image shows how to create flower images (*S,P,J_1,J_2,n*) when *n* is even and one of the jumps is *n*/2 and *P* < S without using the Center-point model. More complex images occur when *P* > *S* and if one of the jumps is not *n*/2.



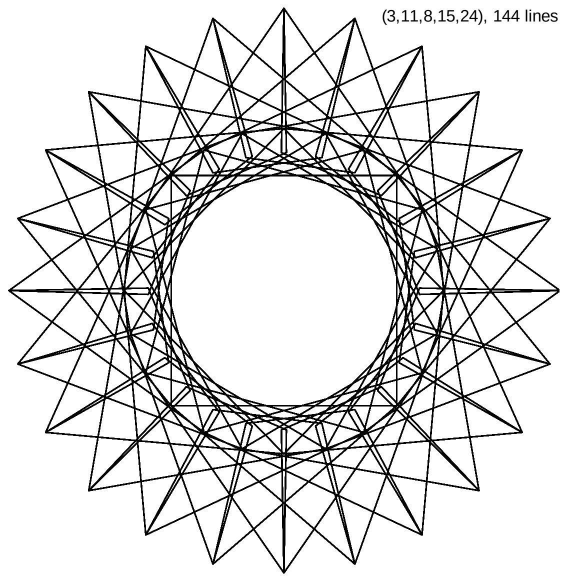
(19,11,1,12,24), 912 lines



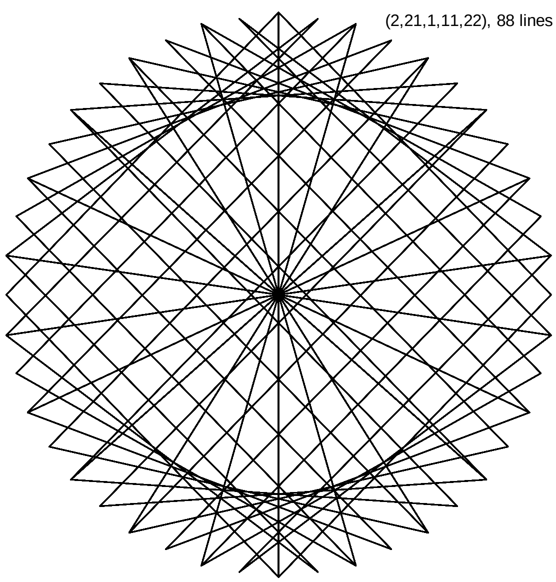
(19,11,1,10,24), 912 lines



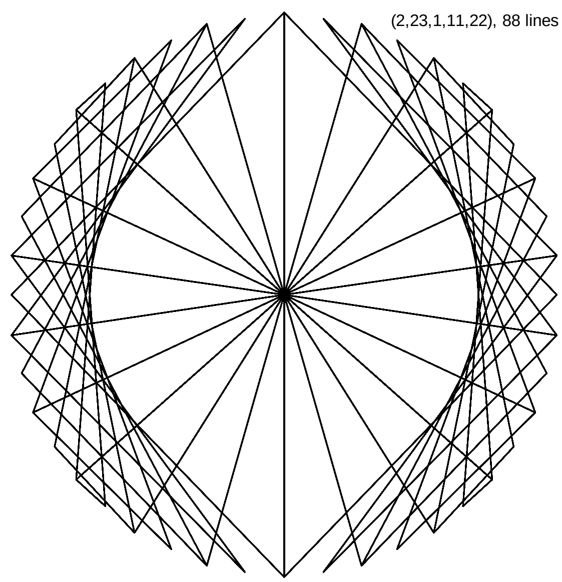
(3,11,2,15,24)



(3,11,8,15,24), 144 lines

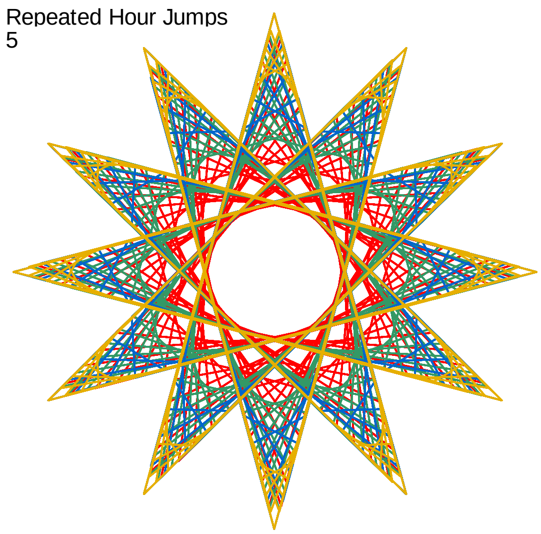


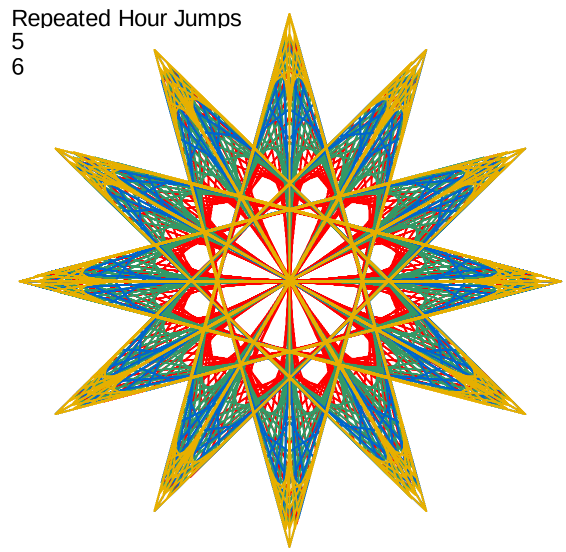
(2,21,1,11,22), 88 lines

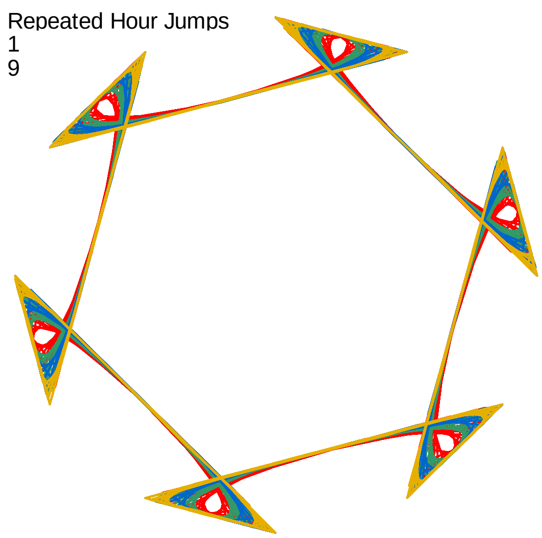


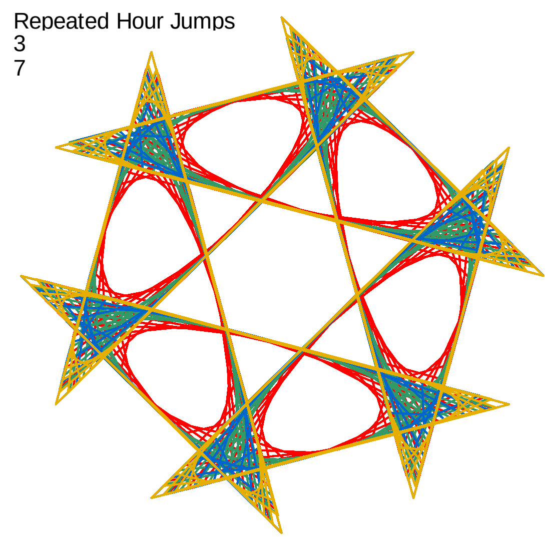
(2,23,1,11,22), 88 lines

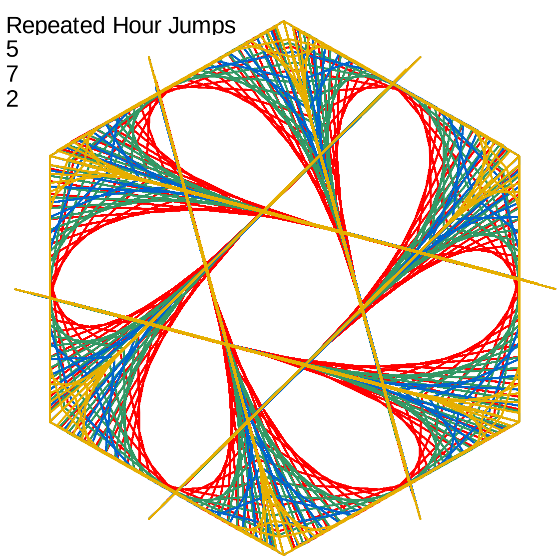
5. Four color model with 1 to 3 jump patterns on dodecagon (an extension of E&E Bridges 2020 article) The images below provide a glimpse into the wide array of images that can be examined using this model. The model is restricted to a dodecagon because it makes it is easy to talk about the vertices as hours on the clock. The first is 1 jump the next 3 are 2 jumps and the bottom 2 are 3 jump models. The hexagon with spikes on bottom left is due to 5+7 = 12 and then jump two vertices over. Had the last value been 3 or 4, a spiked square or triangle results.

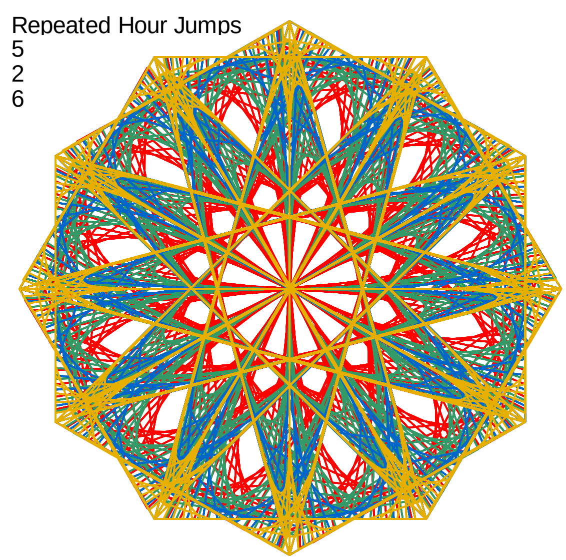






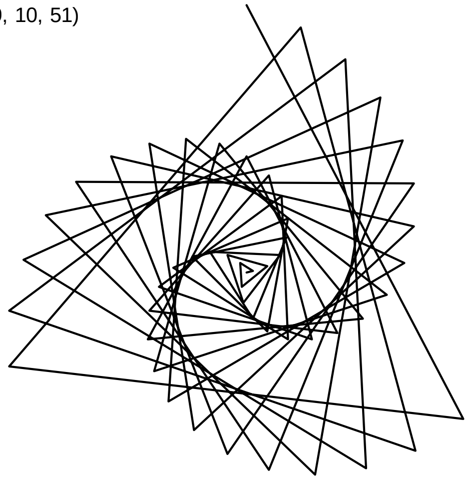




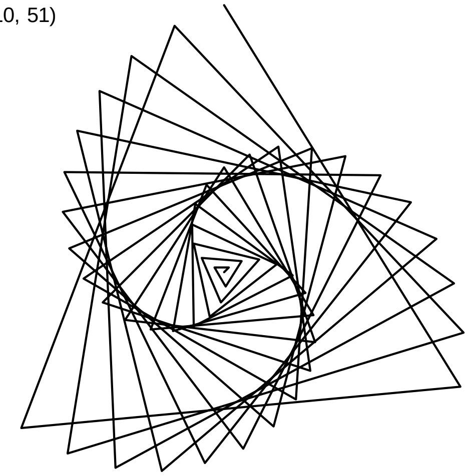


6. Spirals and more (based on Erfle, “Using Archimedean Spirals to Explore Fractions”). A modest addition to File 1 produces spirals. Interesting patterns emerge based on jump pattern, the size of the polygon as well as the reduction in radius, r, per jump. images shown are with (*n, J, r*) labels. Rotating images occur when n/J is “close” to an integer and that image rotates in different directions based on whether the fraction is just over or just under an integer value (*n/J* = 2.9 vs. 3.1 in the top two images).

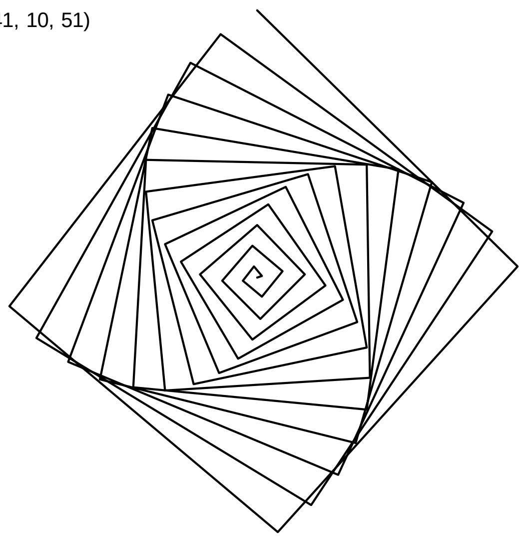
(29, 10, 51)



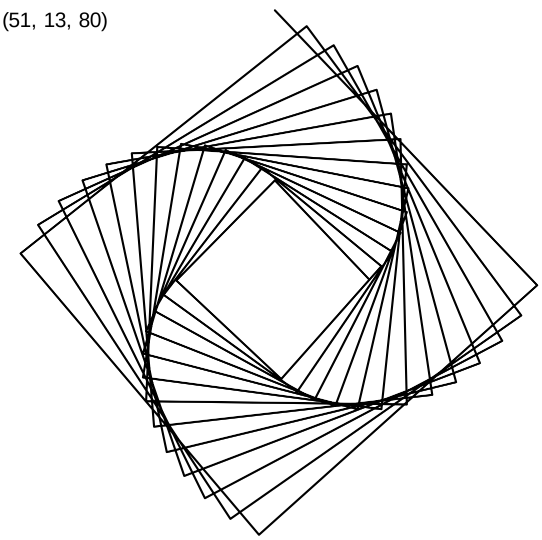
(31, 10, 51)



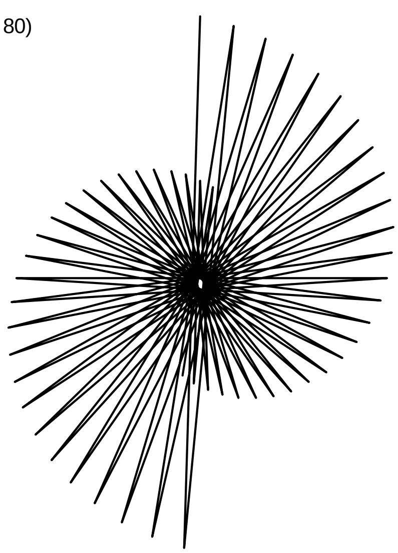
(41, 10, 51)



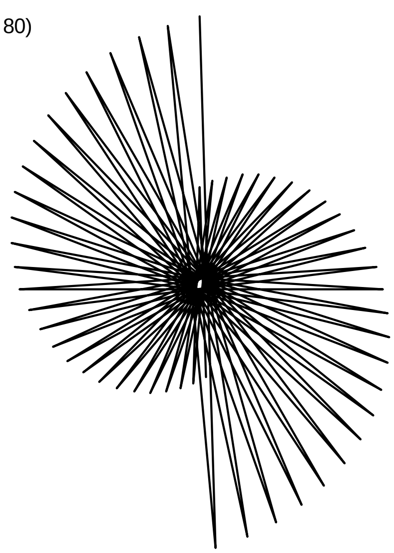
(51, 13, 80)



(49, 25, 80)



(51, 25, 80)

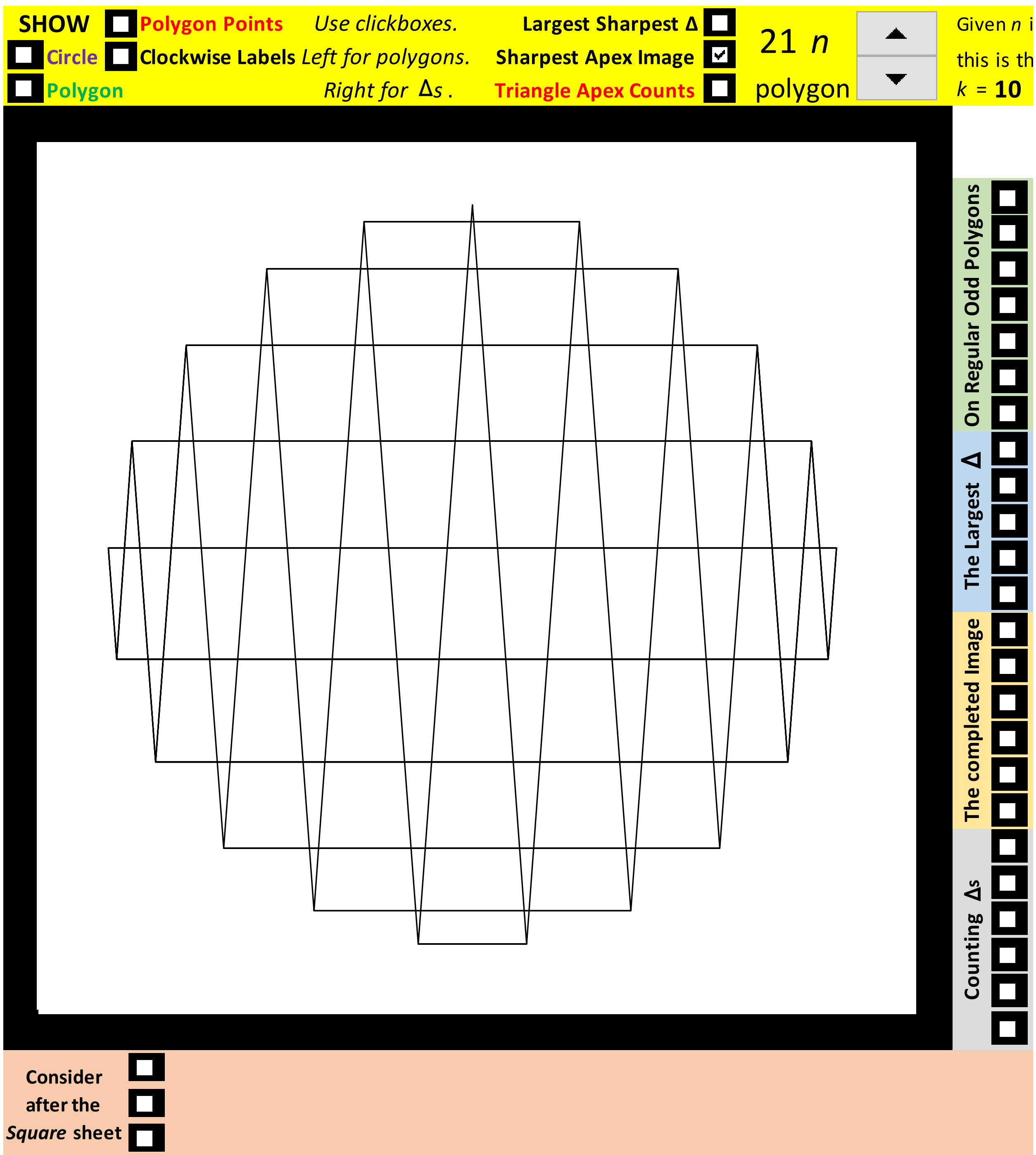


7. Triangles and perfect squares (based on E&C Alternative Visions of Perfect Squares, grades 3-5)

Abstract from paper: Once students know multiplication, they tend to have a firmer grasp on perfect squares than other multiples. This paper keys off that knowledge and shows multiple geometric and numeric patterns in what comprises a perfect square. The accompanying Excel file is a self-contained teaching tool for in-person and remote classrooms.

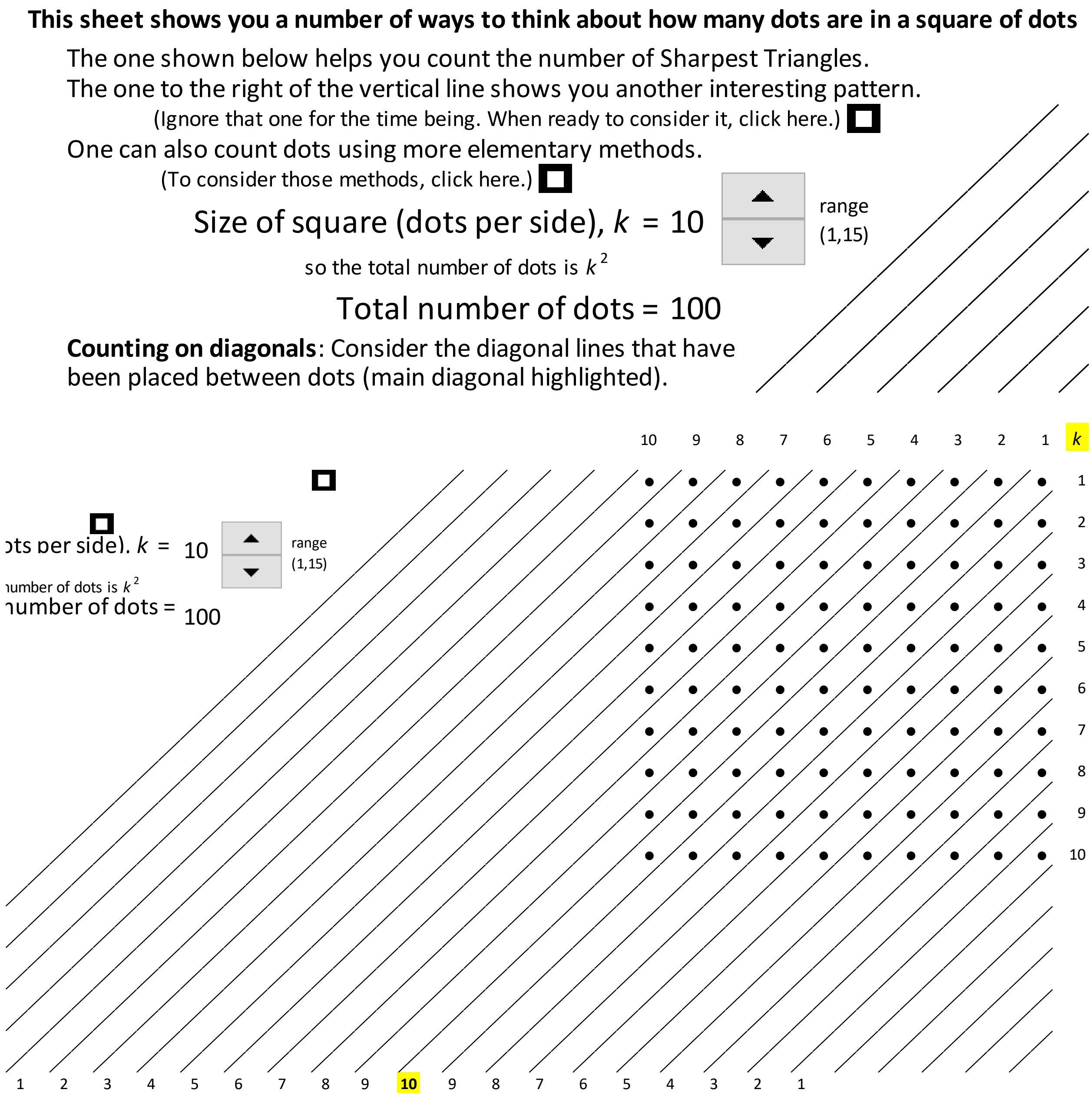
The figure below shows an image with a large number of sharply pointed triangles. An interesting question is: How many triangles are there in the image? The almost magical answer comes by looking elsewhere. The image on the next page shows a perfect square of dots, looked at in a different way. There you see that 1+2+…+9+10+9+…+2+1 = 10^2 = 100. Now look at the left hand side of the image below and count triangle peaks looking in a zig-zag fashion. Note that the counts are identical to the "up the hill and back down" pattern with the top vertex having 10 triangles so there are 100 triangles. The same question can be asked for any odd polygon (and the file allows you to see images from 3 to 31).

Creating Sharpest Isosceles Triangles Embedded in Regular Odd Polygons



(From Erfle and Chakerian paper)

Figure 3. Two pieces of the *Square* sheet given k = 10 showing topics and diagonal elements



We can use this same set-up to examine a couple of other important formulas. The first is a second way to see k^2 in the resulting image. It is based on the L-shaped additions when k increases by 1. Each addition adds $2k$-1 dots. This is the k^{th} odd number. Put another way, the sum of the first k odd numbers is k^2 (see Figure 4 below for further discussion). Additionally, we can see that the sum of the first k numbers is $k\cdot(k+1)/2$ (see Table 2 below for further discussion). These formulas are useful in counting different, more complex, images. These are the kinds of images examined in an open-ended fashion using the general triangular model.

As noted in the abstract, the file is set up to be a self-contained teaching tool. The two tables show the click-able discussion points and questions that are contained on the two main sheets.

Table 1. Clickable *Sharpest Triangles* discussion points and questions in five topic areas

On Regular Odd Polygons	In this figure, each regular odd polygon has a fixed vertex at the top of the circle (click A4 on and I2-I4 off, and use the scroll arrows). Label this top point both 0 and *n*, and other points just like a clock that has *n* hours rather than 12 (click C3 on). This means that the bottom of the polygon will be flat and there is vertical symmetry. This symmetry means that there are *k* paired vertices at the same height in the (*x, y*) plane. In this instance, all vertices from 1 to *k* have a horizontal counterpart in vertices *n*-1 to *n-k* = *k*+1 (click C2 and C3 on and A4 off). Put another way, we could draw *k* horizontal line segments between pairs of vertices. The same thing is true in other directions except that then the *k* parallel lines are no longer horizontal.
The Largest ∆	Three non-parallel lines can be used to create triangular images. We are interested in looking at vertices which create the sharpest apex angle using these vertices. We are also going to restrict ourselves to isosceles triangles (click W2 on to see apex and base angles). Consider the largest such sharpest angle isosceles triangle having a horizontal base and slanted legs. It will have vertices 0, *k*, and *k*+1 -- this is the triangle shown by clicking I2 on (this triangle shares a common base with the polygon).
The completed Image	To create the final image, we connect all other vertex pairs which have lines parallel to each of these three lines. To do this, we must draw the other *k*-1 parallel lines in these three directions. Once done, the completed image emerges (click I3 on). Scroll *n* from 3 to 31 and watch how the images develop (click Circle, Polygon, and Polygon Points off for sharpest image). How would you describe what happens each time *n* increases? The apex angle gets a bit sharper and a new "fold" or "wave" happens with the largest horizontal line just above or below the middle. Notice that the new fold is downward pointing when *k* is odd and upward pointing when *k* is even.
Counting ∆s	We wish to count all triangles of various sizes in this image. Call this number of triangles *T*(*n*). Move from 3 to 5 to 7 and see what happens (remember, some triangles are "upside down") then think about larger *n*. There are various ways to do this, but the easiest is to use apex vertices (since all apex vertices are also the polygon's vertices). Therefore, for each polygon vertex we can attach a count of triangles with apex at that point (click I4 on). Notice that as we move toward the side from the top or bottom, apex counts decline by 2 per vertex ... to see why, focus on bases. There are various ways to sum apex counts around the circle, go to the **Square** sheet to see one elegant method.
After Square Sheet	If *n* is large, and you decide to add numbers of apex counts starting at the top and going clockwise, it will soon become tedious. Instead, start at one of the two vertices with apex counts of 0 located at the end of the wave and follow the zig-zag pattern from one side to the other. Notice the number pattern. From here, *T*(*n*) should be clear. It turns out that there is another interesting way to visualize k^2 using something called gnomons. To read about that, go back to the **Square** sheet and click the box in AA4. Finally, click Q6 for two additional methods.

Figure 4. Partial *Square* sheet given k = 10 discussing gnomons

k	gnomon	**Now, ignore the diagonal lines. Instead, focus on the ∟ shaped dots that are added to the left and on the bottom as k increases.**
		The nomenclature for this "L" shaped addition is gnomon.
1	1	To count the gnomon of dots, note that adding a row adds k dots, and adding a column adds k dots, and one of those dots,
2	3	the bottom left corner, is common to both row and column. Each gnomon is therefore of the form 2k-1, where 2k-1
3	5	is the kth odd number. This means that: The sequence of gnomons is the sequence of odd numbers.
4	7	In other words, the sum of the
5	9	first k odd numbers is k^2.
6	11	
7	13	
8	15	
9	17	
10	19	
	100	= sum of the first 10 odd numbers.

Table 2. Clickable notes at the bottom of the *Square* sheet showing another use for the hill formula: Gauss addition

1 The hill formula provides a 'side door' to an even more famous pattern in numbers formula: Suppose you are asked to sum the numbers from 1 to 100?

2 If we increase k to 100, the hill pattern would have that sum plus the sum from 99 to 1. If we add 100, we have twice the sum from 1 to 100. Therefore, $100^2 + 100$ = twice the sum from 1 to 100.

3 Dividing by two we have: The sum from 1 to 100 = $(100^2 + 100)/2 = 100\cdot(100+1)/2 = 5{,}050$.

4 More generally, $1 + 2 + \ldots + k = k\cdot(k + 1)/2$. This is an example of Gauss addition (click next box to learn more).

<u>The highlighted material may be too difficult for Grades 3 and 4.</u>

5 The classic story goes that Carl Friedrich Gauss recognized a pattern as a young child when asked to sum the numbers from 1 to 100.

6 He noticed that if you take a second copy of those numbers and reverse their order and put them on top of one another, something magical occurs. To see this click the next box.

7

$$\begin{array}{r} 1 + \quad 2 + \quad 3 + \ldots + \quad 98 + \quad 99 + 100 \\ 100 + \quad 99 + \quad 98 + \ldots + \quad 3 + \quad 2 + \quad 1 \\ \hline \end{array}$$

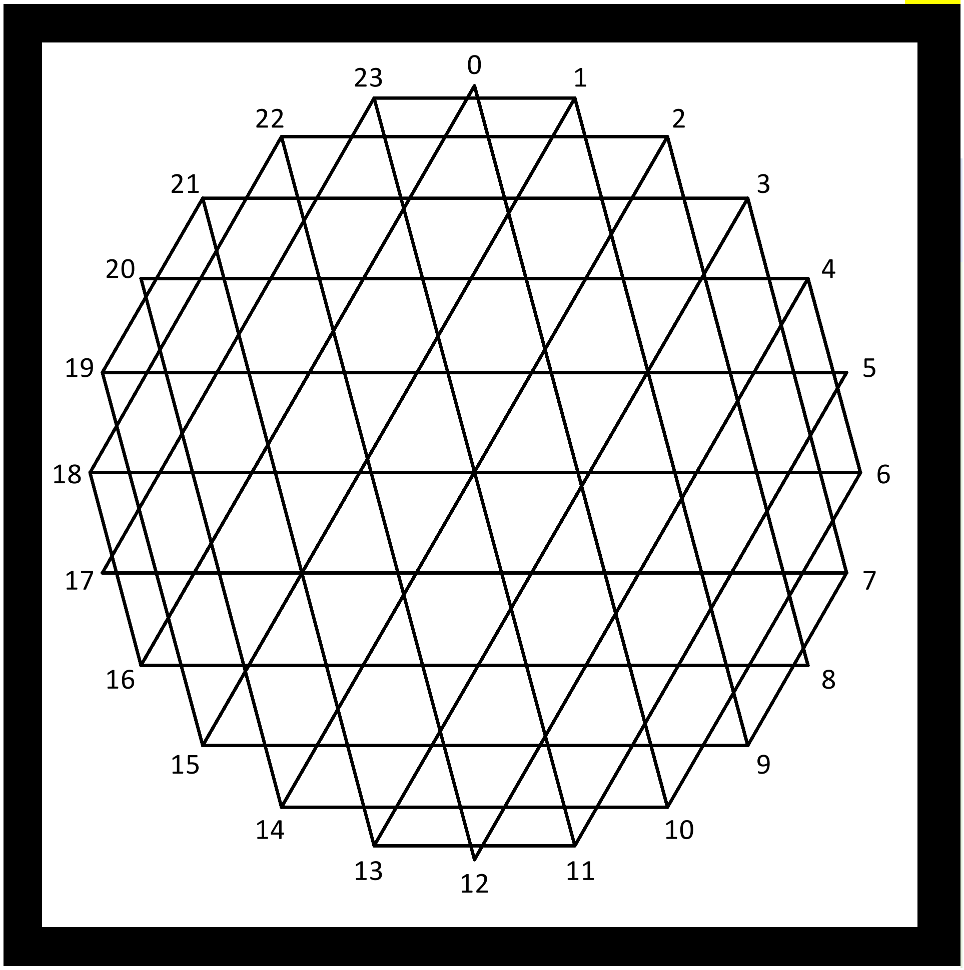
Instead of adding horizontally, add vertically: $101 + 101 + 101 + \ldots + 101 + 101 + 101$

8 Each vertical sum is the same and the top row shows how many 101s are present. Therefore, twice the sum of 1 to 100 is $100\cdot101$ so the sum of 1 to 100 = $100\cdot101/2 = 5{,}050$.

9 It is worth noting that it is standard practice to go in the opposite direction and derive the hill formula (1) from the sum of the first k numbers formula (3) rather than deriving (3) from (1).

8. General triangular model sets up triangular images connecting polygonal vertices in three directions. The following is the dashboard with instructions for using the file.

Create your own triangular patterns on regular polygons, *n* 24



(2 < *n* < 32) **Images are created by defining three non-parallel lines between vertices**

Vertex #'s ☑ ☐ Points **For simplicity the first two lines use vertex 0 = (0, 1).**

SHOW Circle ☐ **Line 1 is 0 to *j* = 16 *j* = 1, ..., *n*-1. Line 2 is 0 to *k* = 10 *k* ≠ *j*, *k* = 1, ..., *n*-1.**

Lines ☑ Line 3: 1st vertex, *v* = 4 *v* = 0, ..., *n*-1. 2nd vertex, *w* = 20 *w* ≠ *v*, *w* = 0, ..., *n*-1.

Parameter values ☐ ***Note:* The third line need not include *j* or *k*, although that is fine.**

a, b, c are arcs of circle summing to *n* and represent angles *a/n* ·180°, *b/n* ·180°, and *c/n* ·180° — 6 = *a* = min(|*j*-*k*|,|*j*-*s*|,|*s*-*k*|)

To obtain *a*, *b*, & c from *j*, *k*, *v*, *w*, & *n*: — 0 = *s*, The line *vw* is parallel to 0*s* with *s* = MOD(*w*+*v*,*n*) — 10 = *b* = *MAX* - *a*

16 = *MAX* = MAX(|*j*-*k*|,|*j*-*s*|,|*s*-*k*|) — 8 = *c* = *n* - *b* - *a*

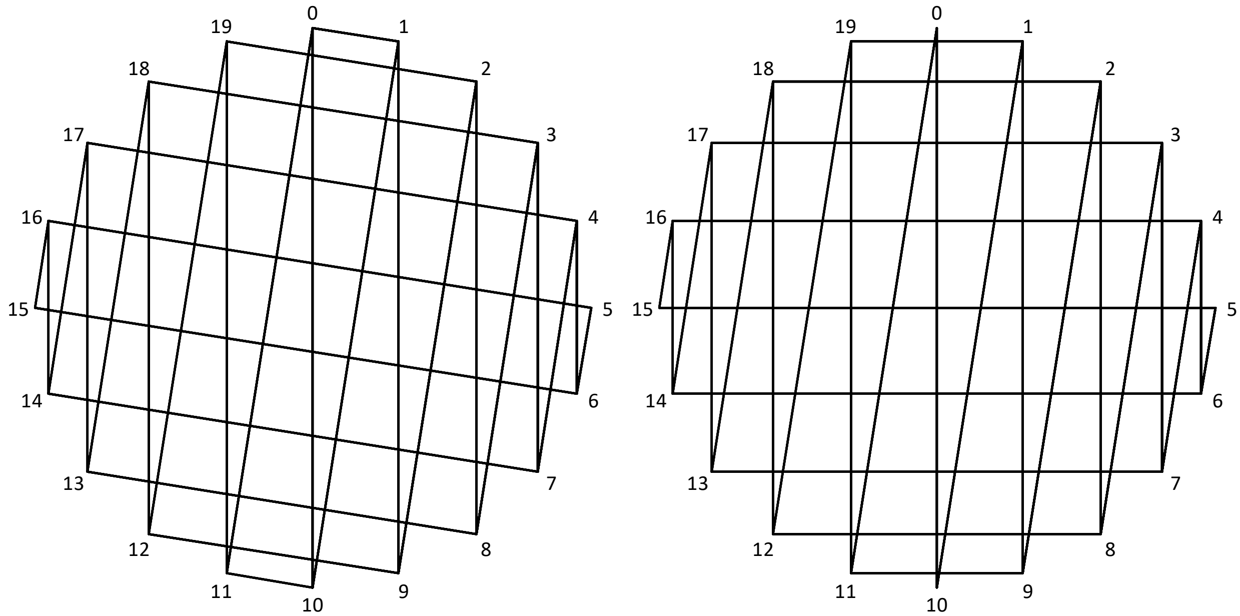
Additional instructions: You can manually enter numbers in the yellow cells, or you can insert equations in those cells. The yellow cells have been labeled so you can refer to them by name. For example entering, = INT(*n*/3) in Q3, = *n*-*j* in V3, 1 in R4, & = *n*-*v* in W4, produces "near equilateral" isosceles Δs. These are exact when *n* is divisible by 3.

To rotate image: 0 = *r*, rotation factor r = 0, 1, ..., *n*-1 using these equations
j = MOD(j_0+2r,n), k = MOD(k_0+2r,n), v= v_0, w = MOD(w_0+2r,n)

To create parallellograms, set *v* = 0 and *w* = *j*.

This file can be used to examine a large array of images. This provides for open-ended exploration based on various rules. For example, the previous chapter images of sharpest angle isosceles triangles using odd polygonal vertices are obtained with odd *n* = 2*k*+1 with *j* = *k*+1, *k* = *k*, *v* = *j*, and *w* = *k* (the image shown in part 4 above set *n* = 21, thus, *k* = 10). Beyond that, we might wonder how things change when *n* is even, or when the triangles are not isosceles. Both those questions can be answered by creating and analysing images using this file.

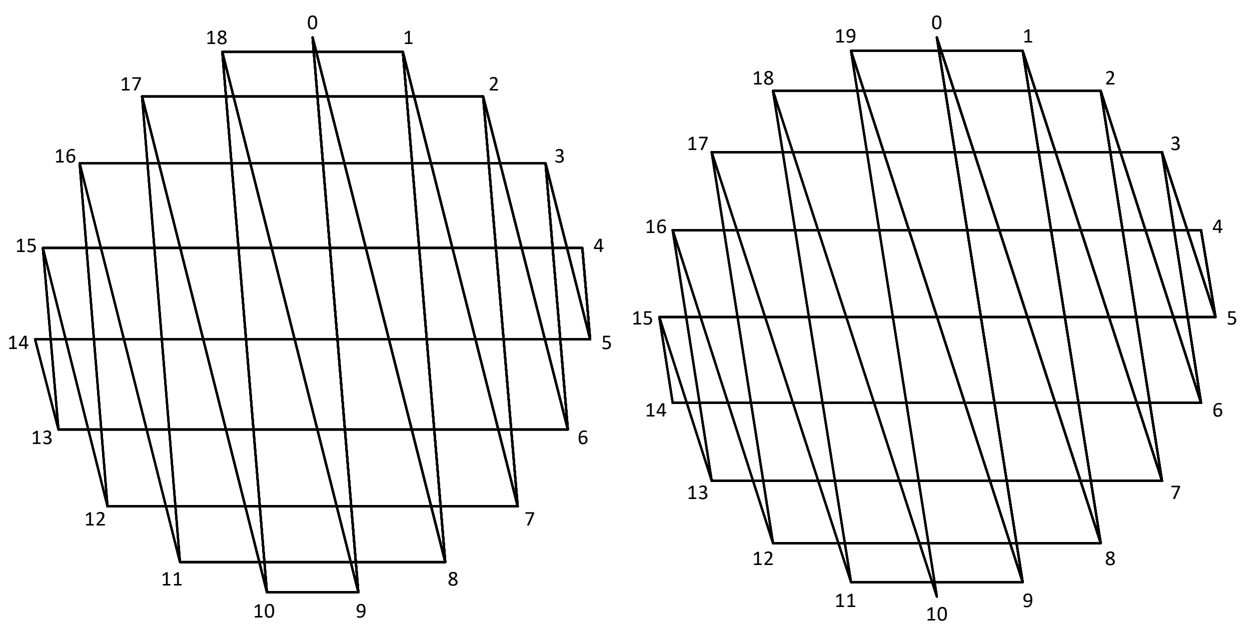
9. The general case of counting sharpest triangles. (Images from sharpest apex triangles paper)
When n is even, a sharpest apex triangle (in which two of the vertices chosen are consecutive to one another) cannot be isosceles since j+ (j+1) is, by necessity, an odd number. But, when $j = n/2$ and $k = j+1$ the resulting triangles are tall-skinny right triangles like those shown in the left panel below. The right panel shows the same apex angle given horizontal rather than slanted bases. It turns out that both have the same apex pattern count of $1+2+\ldots+8+9 + 9+8+\ldots+2+1 = 9\cdot10 = 90$ triangles.



(a) Slanted base (b) Horizontal base

Figure 1. Two versions of sharpest apex right triangles

For the more general situation with angles $(1, b, c)\cdot(180/19)°$ (where $b + c = n - 1$) and $b \neq c$, there will be a smaller number between b and c (let that be b). The zig-zag wave pattern is still visible, but it now plateaus with apex counts of b. The total number of triangles is $T(n, b) = b\cdot(b + 1) + b\cdot(n - 2\cdot(b + 1))$. The triangle count is therefore 80 in Figure 2.a and 88 in Figure 2.b.



(a) $n = 19$, $(1, 8, 10)\cdot(180/19)°$ (b) $n = 20$, $(1, 8, 11)\cdot9°$

Figure 2. Two versions of sharpest apex obtuse triangles inscribed on n-gons with horizontal base