

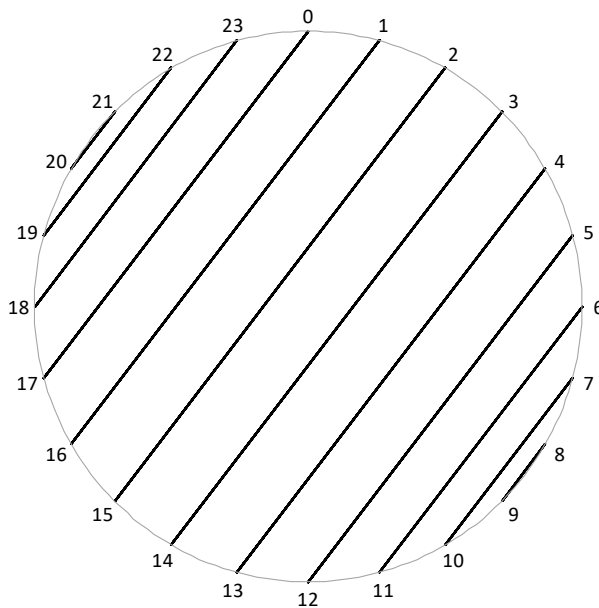
Creating images of triangles inscribed using the vertices from regular polygons

Stephen Erfle, January 2, 2021

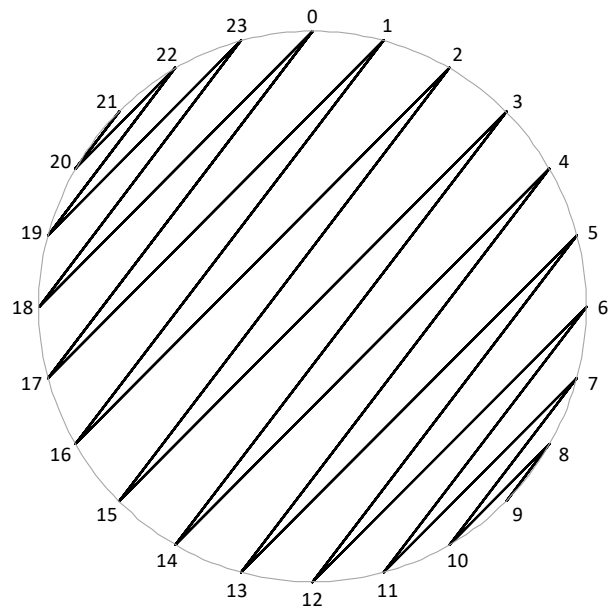
The vertices of regular polygons are equally spaced around a circle. If the polygon has n sides and vertices (we will often call this an n -gon) then the central angle of a regular n -gon is $(360/n)^\circ$ because each of the n arcs is equal sized, and they must sum to 360° . This equal spacing provides the opportunity to use polygonal vertices to create images having parallel lines. We first turn to how to create parallel lines using polygonal vertices and why those lines are parallel.

Parallel lines. To create a set of parallel lines we must first decide on the size of the polygon and a pair of vertices that will form the endpoints of the line segment. Let $n = 24$ and choose vertices 2 and 15. Figure 1.a shows the set of lines parallel to the line 2-15. (Vertices are counted clockwise starting with 0 at the top, and v_1-v_2 signifies the same line segment as v_2-v_1 because the order vertices are listed does not matter.) If we increase one vertex by 1 and decrease the other by 1, one of two new lines result, 1-16 or 3-14. These lines are parallel because of two theorems from elementary geometry: the inscribed angle theorem; and the alternate interior angle theorem. The former says that any inscribed angle on a circle is half the size of the central angle. The latter says that if a line transverses two lines and if the alternate interior angles are identical, then the two lines are parallel.

To be explicit, Figure 1.b shows the four angles created by the lines from 2-16 and 3-15: one is from 16 to 2 to 15, call this $\angle 2$; together with $\angle 3$, $\angle 15$, and $\angle 16$. These four angles, together with the 18 other angles at all vertices except 9 and 21, are identical because of the inscribed angle theorem. Indeed, each angle is $7.5^\circ = \frac{1}{2}(360/24)^\circ$. Since $\angle 2$ is identical to $\angle 16$, 1-16 is parallel to 2-15, and since $\angle 3$ is identical to $\angle 15$, 3-14 is also parallel to 2-15 by the alternate interior angle theorem. All other lines shown in 1.a are parallel to one another in the 2-15 direction (and for similar reasons, the second set of lines created in 1.b are parallel to 2-16). It is worth noting that although the lines are parallel to one another, they are not equidistant from one another.



(a) A set of lines based on the line 2-15



(b) Sharpest angle created using next closest vertices

Figure 1. Using 24-gon vertices to create sharpest angle and two sets of parallel lines

Creating triangular images. Triangles are created whenever three non-parallel lines intersect in a pairwise fashion. If the three lines intersect at a single point, the triangle is degenerate, otherwise, the triangle has three sides, three angles and three distinct vertices. The image in Figure 2 has three such interior concurrences (all of which are on the line (not shown) between 4 and 16) and many triangles of various sizes. This image is shown in order to describe the various aspects of this Excel file. The file is based on five parameters: n , j , k , v , and w , and 5 checkboxes that control attributes of the final image produced. If you would like to create your own image(s) using pencil, ruler, and paper, you can create worksheets with one or more polygon vertex images by deciding what value(s) of n you wish to use as well as whether you want to provide labels then follow the instructions provided in A31.

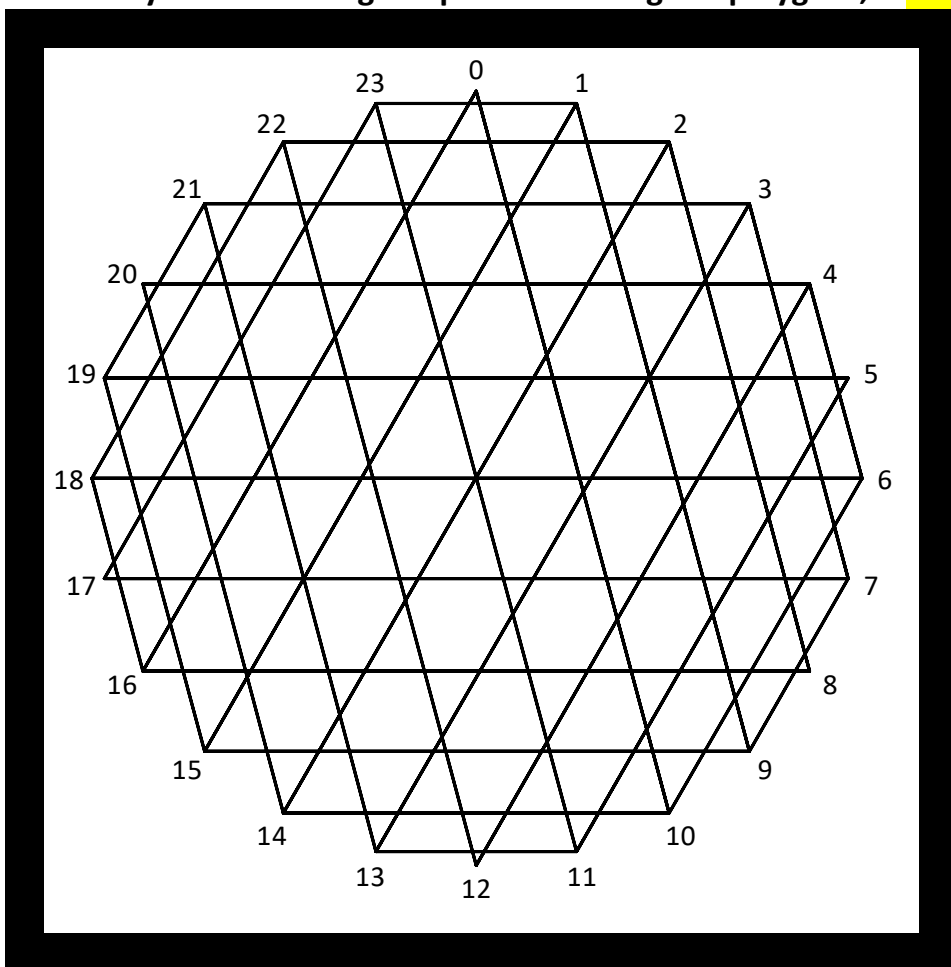
Each polygonal vertex includes either two or three lines. Therefore, we can use vertex 0 as the starting point for two of the lines. The other endpoints are at j and k (highlighted in yellow in Figure 2). The third line is $v-w$ but, of course, many (v, w) pairs would produce the same image, as all would be in the same direction. In the image shown, the third line is horizontal, and the same image occurs for all other values where $v + w = 24$. If $v = w$, the system of parallel lines which excludes v is created (so that the same image would have occurred, had $v = w = 0$ or $v = w = 12$ given $n = 24$).

Similar triangles. Similar triangles have the same angles but not necessarily the same length sides. All triangles in each image created using this file are similar due to the parallel nature of each of the three lines to one another. The angles in triangles created on regular n -gons can only take on certain values because they must be a multiple of $(180/n)^\circ$, the smallest possible angle using three distinct vertices of an n -gon (shown above in Figure 1.b as 7.5° given $n = 24$). The multiple is visually obtained by counting the number of unit sized arcs formed from a vertex on the polygon to where the angle hits on the other side of the image. To take one example, the sharpest angle in Figure 2 is most easily described as $\angle 0$ or $\angle 12$ (because each has only two lines at those vertices) but perhaps the easiest solution is to note that it is the same angle as the one that is formed from the lines 0-10 and 10-6. This angle is $6/24 \cdot 180^\circ = 45^\circ$. The multiples a , b , and c , that sum to n can also be obtained algebraically from n , j , k , v , and w . This requires a bit of manipulation which it is not necessary to replicate here (or to understand for that matter). The blue highlighted area at the bottom of Figure 2 shows these calculations.

The number of lines per vertex and the number of parallel lines. As noted above, the image is constructed using 3 non-parallel lines connecting pairs of vertices. All other lines in the image are parallel to one of these three lines. These parallel lines are generated by moving one polygonal vertex clockwise and one counterclockwise from an initial set of vertices.

For example, consider the line 6-16 in Figure 3.a. The two closest parallel lines are 5-17 and 7-15. Others are further away from the initial line but there is an interesting rule at work; the sum of the vertices is 22 (or 2, the remainder when the sum is divided by $n = 20$). This works for the other two directions as well, the remainder when the sum of vertices is divided by n is constant. The horizontal line sums to $20 = n$ and the third line sums to 18. Notice the importance of considering both sums by looking at lines parallel to 0-17 in 3.b. All parallel lines below 0-17 sum to 17 but 18-19 sums to 37 which has remainder 17 upon division by 20. (Another term for the remainder function is MOD, short for modulus in modular arithmetic. This is the Excel function used to define the endpoint s of line 0- s parallel to $v-w$ in the blue highlighted area at the bottom of Figure 2.)

Create your own triangular patterns on regular polygons, n 24



▲ (2 < n < 32) Images are created by defining three non-parallel lines between vertices

▼ Vertex #'s Points For simplicity the first two lines use vertex 0 = (0, 1).

SHOW Circle Line 1 is 0 to j = 16 j = 1, ..., n-1. Line 2 is 0 to k = 10 k ≠ j, k = 1, ..., n-1.

Lines Line 3: 1st vertex, v = 4 v = 0, ..., n-1. 2nd vertex, w = 20 w ≠ v, w = 0, ..., n-1.

Parameter values Note: The third line need not include j or k, although that is fine.

a, b, c are arcs of circle summing to n and represent angles $a/n \cdot 180^\circ, b/n \cdot 180^\circ,$ and $c/n \cdot 180^\circ$ $6 = a = \min(|j-k|, |j-s|, |s-k|)$

To obtain $a, b, & c$ $0 = s, \text{ The line } vw \text{ is parallel to } 0s \text{ with } s = \text{MOD}(w+v, n)$ $10 = b = \text{MAX} - a$

from $j, k, v, w, & n$: $16 = \text{MAX} = \text{MAX}(|j-k|, |j-s|, |s-k|)$ $8 = c = n - b - a$

Additional instructions: You can manually enter numbers in the yellow cells, or you can insert equations in those cells.

The yellow cells have been labeled so you can refer to them by name. For example entering, = INT(n/3) in Q3, = n-j in V3,

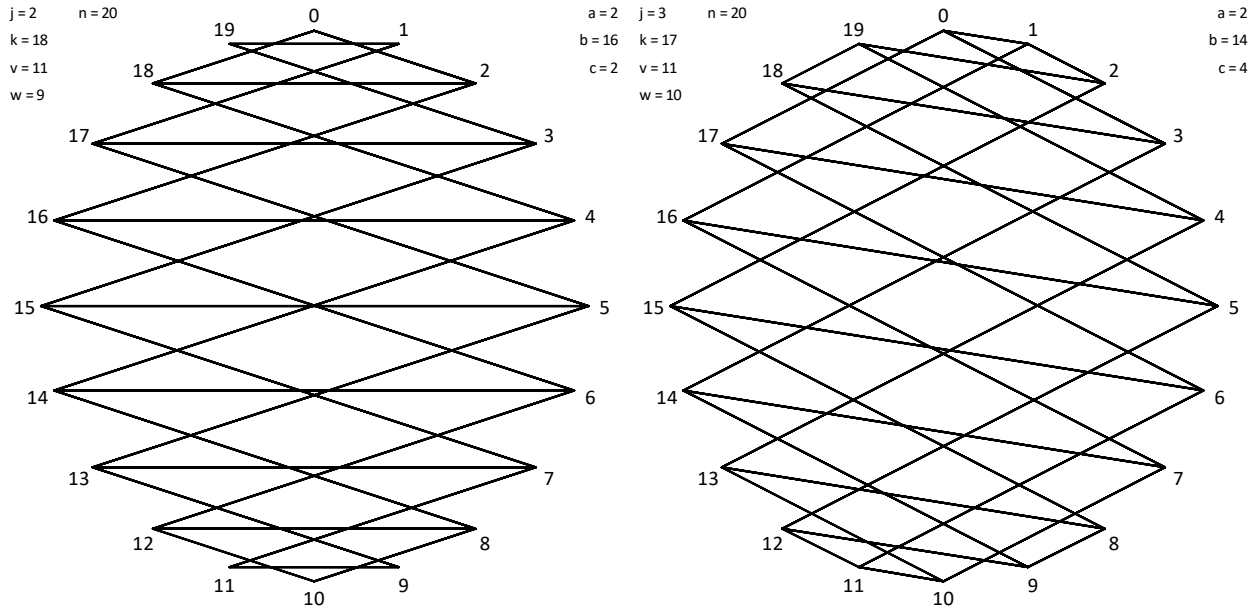
To rotate image: 1 in R4, & = n-v in W4, produces "near equilateral" isosceles Δ s. These are exact when n is divisible by 3.

0 = r, rotation factor r = 0, 1, ..., n-1 using these equations

$j = \text{MOD}(j_0+2r, n), k = \text{MOD}(k_0+2r, n), v = v_0, w = \text{MOD}(w_0+2r, n)$ To create parallelograms, set v = 0 and w = j.

Figure 2. Images from *Triangles on Regular Polygons* file with dashboard

The two images in Figure 3 show an interesting contrast. In 3.a, the top and bottom three vertices have two lines, and all others have 3, but in 3.b, all vertices have 3 lines. It is worthwhile to examine the rules that determine how many polygonal vertices have a full complement of the lines and how many have two. In the process, we will learn how many parallel lines are in each direction.



(a) An image with 6 two-line vertices

(b) An image with no two-line vertices

Figure 3. Two images using 20-gon vertices to create a similar triangles image

The easiest way to see why a polygonal vertex is excluded from having a line in a direction is that no other vertex is available in that direction. Take, for example, vertex 1 in 3.a. The segment 0-2 is the smallest such segment in that direction. To see why, move down one line to 19-3. This line has 4 arc-segments and three vertices between vertices 19 and 3 counting clockwise ($19 \frown 20 + 0 \frown 3$ where $v_1 \frown v_2$ signifies the arc starting at v_1 and ending at v_2) meaning that a parallel line inside this arc, 0-2, is possible. The reason 0-2 cannot have a parallel line inside that arc is that there is only one vertex inside $0 \frown 2$, namely 1.

When the number of arc-segments is even, as is true on both sides of the line in all three directions in Figure 3.a, there are two vertices in each direction having two lines. At the opposite end of the spectrum, if the number of arc-segments is odd on both sides of each line, then no vertices have two lines, just like in 3.b.

The easiest way to find the number of arc-segments in $v_1 \frown v_2$ is to subtract $v_2 - v_1$ if $v_1 < v_2$ or $n - |v_2 - v_1|$ if $v_1 > v_2$. Consider again the segment from 19-3 in Figure 3.a. There are two arcs, $3 \frown 19$, and $19 \frown 3$. The first has $19 - 3 = 16$ arcs which is even, thus there will be a vertex between 4 and 18 which excludes a parallel to 3-19. Indeed, we know this vertex is the midpoint of $3 \frown 19$, or average of 19 and 3 which is 11 in Figure 3.a. The second arc includes the peak apex which we have labelled 0 (but which we could also have labelled 20 or n). Counting arcs which include this vertex is $(20 - 19) + 3 = 4$ arcs as noted above. (Note that the two arcs comprise the circle, thus if you know the size of one arc, then subtract that from n to find the size of the other arc.) The midpoint is readily seen by imagining that this is a 20-hour clock and the midpoint between 19 and 3 is 1. (Alternatively, note that once you know one midpoint, the other is $n/2$ vertices away from that midpoint on the other side of the circle.)

The same midpoint calculations work when the number of arc-segments is odd. In this instance the midpoint will be midway between two numbers, the two numbers which form the smallest line in that

direction. Consider the segment 15-2 in Figure 3.b. One arc, $2 \frown 15$, has midpoint at $(2 + 15)/2 = 8.5$ midway between 8 and 9 which corresponds to the smallest segment on the $2 \frown 15$ arc. On the other side is the arc, $15 \frown 2$ has a midpoint of $(15 + (20 + 2))/2 = 18.5$ midway between 18 and 19 which corresponds to the smallest segment on the $15 \frown 2$ arc.

The rules thus far appear to show that there are either 0 or 2 two-line vertices per direction, but that is because n in Figure 3 happens to be even. When n is even, there will be 0, 2, 4, or 6 two-line vertices in the resulting image. When n is odd, there will always be 3 two-line vertices, one in each direction because the sum of the two arcs must be n . If one of those arcs is even, then the other must be odd, and vice versa. Figure 4 shows parallel images to Figure 3 (same j, k, v , and w); the only difference is $n = 19$ rather than 20. The two-line vertices are 1, 9, and 10 in 4.a, and 1, 11, and 18 in 4.b.

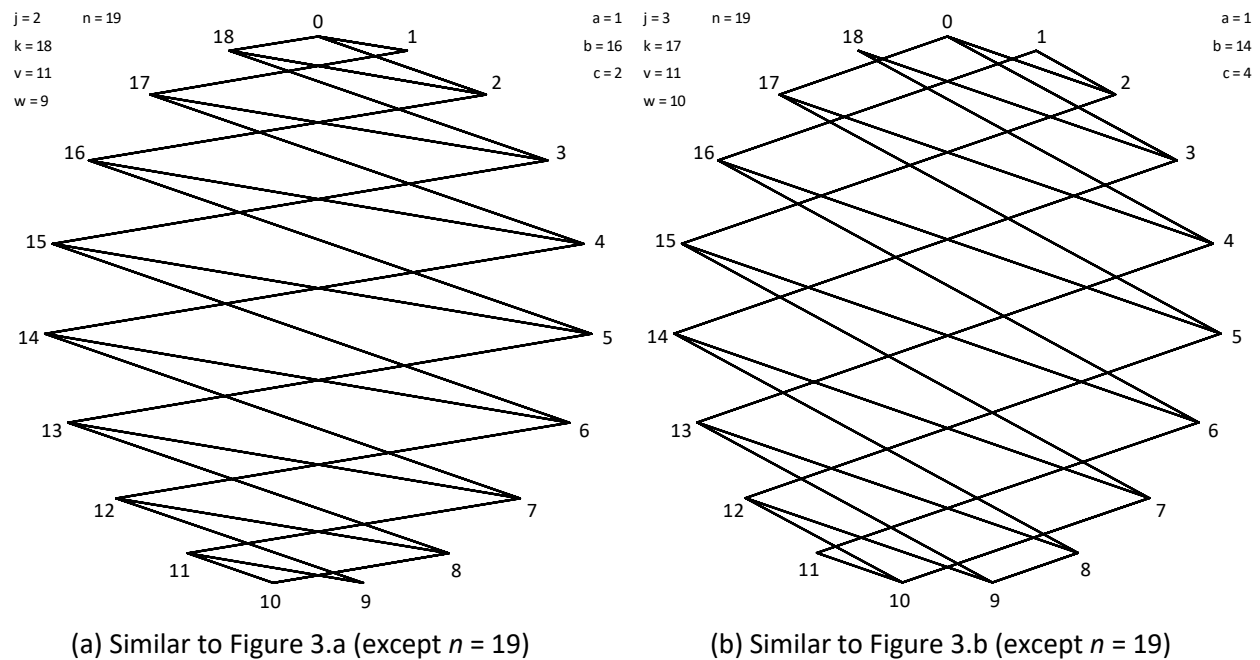


Figure 4. Two examples of odd polygons showing 3 two-line vertices

This naturally leads to the discussion of how many lines are in each direction. When n is odd, the answer is $(n-1)/2$ in each direction because each line in a given direction has two distinct polygonal vertices as endpoints, and, as noted in the discussion of Figure 4, one vertex is left over, thus $n-1$ have vertices that are endpoints to lines in that direction.

When n is even, the answer is: it depends. If, in a given direction, 2 vertices are excluded (as was the case in all three directions in Figure 3.a) then there are $n/2 - 1$ lines in that direction. But if no vertices are excluded in that direction, there are $n/2$ lines in that direction (as was the case in all three directions in Figure 3.b). Put another way, there are three more lines in Figure 3.b (30) than 3.a (27) even though both are based on $n = 20$.

Forms of symmetry. Some of the images above are symmetric and others are not. And some are symmetric in more than one way. Images based on regular polygons may exhibit three types of symmetry: line, point, and rotational. All even polygons will exhibit point symmetry, but odd polygons will not produce point symmetry. This follows from the discussion above regarding the number of two-

line vertices in an image; with even polygons the same number of two-line vertices (0 or 1) are at the midpoint of both arcs of the circle defined by any line from vertex to vertex, while with odd polygons, there is always a two-line vertex at one midpoint but not the other.

Point symmetry can also be conceptualized as rotational symmetry of order 2 because the same image occurs upon rotation by $180^\circ = 360/2$. All images in Figures 1 through 3 look the same when viewed upside down. By contrast, the two images in Figure 4 look different when turned 180° because if either of these images is turned upside down, then the “top” of the image would be midway between vertices 9 and 10 and bottom would be the single point, vertex 0.

The equilateral triangles in Figure 5 show that rotational symmetry need not be 180° . The equal angles in each direction imply that when the image is rotated 120° , or order 3, the same image appears. Equilateral triangles require n be a multiple of 3, such as 9 and 12 in 5.a and 5.b. Additionally, when n is even, as in Figures 5.b and 5.c, the image exhibits rotational symmetry at 60° , or order 6.

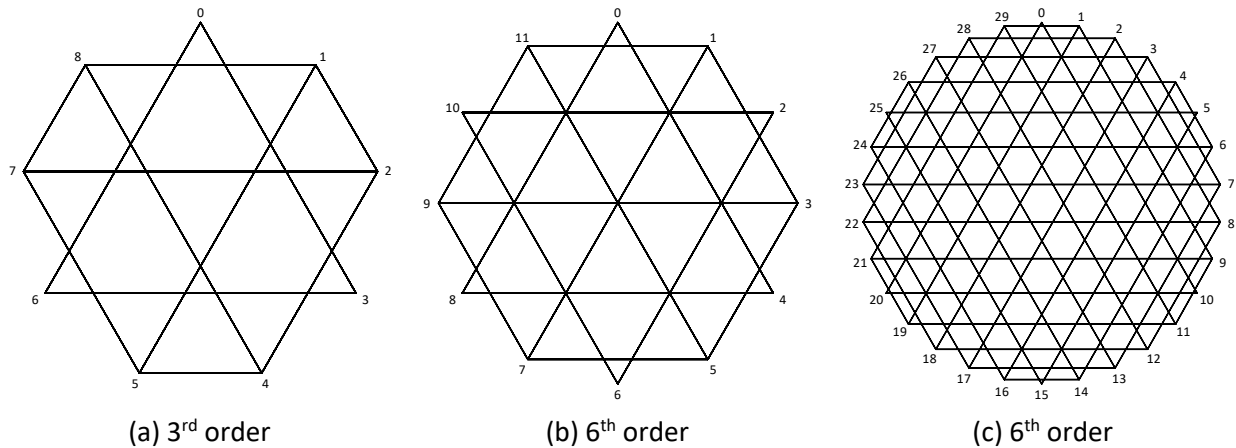
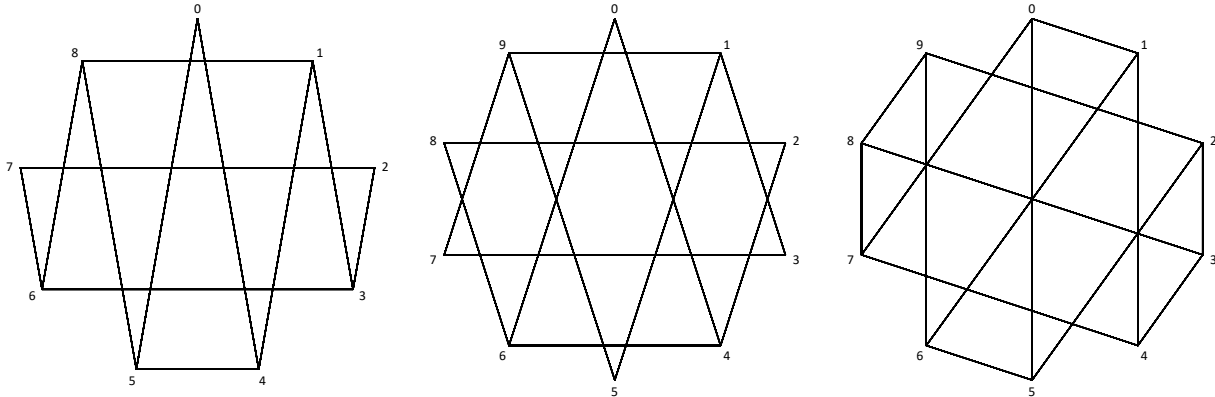


Figure 5. Rotational symmetry from equilateral triangular images from $3m$ -gons for $m = 3, 4,$ and 10

A line of symmetry involves having the image on one side of the line mirrored on the other. Indeed, two other names for this type of symmetry is reflective and mirror symmetry. The (unseen) vertical line from 0- 4.5 in Figure 5.a is one of three lines of symmetry in that figure (the - - between numbers signifies an unseen line). The other two are those from 3- -7.5 and 1.5- -6. Note that each of these lines is perpendicular to one set of parallel lines. The same lines of symmetry exist in 5.b plus three more, the lines formed by the central hexagon 1-7, 3-9, and 5-11. The same six lines of symmetry from 5.b are visible in 5.c at lines $j- -(j+15)$ for $j = 0, 2.5, 5, 7.5, 10,$ and 12.5 . This is the maximum number of lines of symmetry possible given triangular images built on polygonal vertices.

Since lines mirror one another across a line of symmetry, so must angles and vertex points. This implies two things: a triangular image will only have a line of symmetry if the triangles are isosceles (or, as shown above, equilateral); and, a line of symmetry must pass through the center of the image (i.e. it must be a diameter of the circle on which the polygon is inscribed). One line of symmetry must be perpendicular to the lines that form the isosceles triangle base. When n is odd, that is the only line of symmetry in the image. Figure 6.a shows an example of an isosceles triangle with base angles four times the size of the apex angle ($a = 1, b = c = 4$, apex angle 20° and base angles 80°). The vertical line through vertex 0 is a line of symmetry and there is no secondary line of symmetry, since $n = 9$.

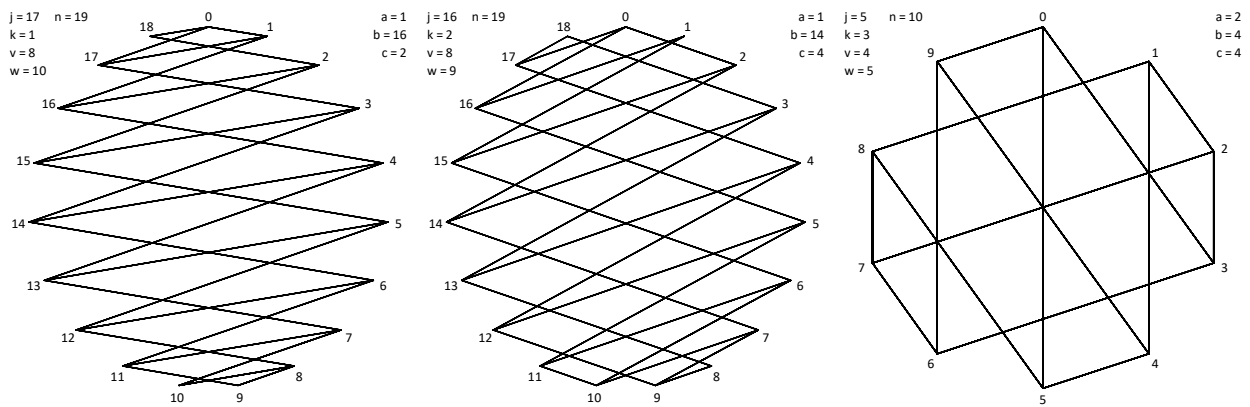
Figure 6.b and 6.c have isosceles triangles with base angles twice the apex angle ($a = 2, b = c = 4$, apex angle 36° and base angles 72°). Figure 6.b has a vertical line of symmetry, 0- -5, but, because n is even, there is a secondary horizontal line of symmetry, 2.5- -7.5. In 6.c, the line of symmetry perpendicular to the base lines is 0.5- -5.5 and the secondary line of symmetry perpendicular to that is 3-8.



(a) 9-gon with 3 two-line vertices (b) 10-gon with 6 two-line vertices (c) 10-gon with 0 two-line vertices

Figure 6. Examples of isosceles triangular images

The rule for vertical reflection. Many of the images we have examined are not vertically reflective, and some have no line of symmetry, so they are not reflective at all. If you want to reflect an image about the 0- $-(n/2)$ axis using the Excel file, a simple rule applies. Consider any image based on the 5 parameters that define the image, n, j, k, v , and w . Consider a new image, $n, n-j, n-k, n-v$, and $n-w$. This image will provide a mirror to the original image, having the same angles but the “reverse” tilt (unless it was already vertically reflective (like Figures 3.a, 5.a, 5.b, 5.c, 6.a, and 6.b)). Three examples are considered in Figure 7, the first two are vertical reflections of the asymmetric images in Figures 4.a and 4.b while the last reflects the symmetric image in 6.c. Each is provided with parameters showing.



(a) Compare with 4.a

(b) Compare with 4.b

(c) Compare with 6.c

Figure 7. Examples of vertically reflected images

Rotating images. Due to the symmetry inherent in Figures 6.c and 7.c, one could imagine transforming one into the other by rotation rather than reflection. For example, rotate 7.c by $36^\circ = 360^\circ/10$ clockwise to obtain 6.c; in the process, vertices 0, 1, ..., 8, 9 from 7.c turn into 1, 2, ... 9, 0 in 6.c. But even when the

image is not symmetric, one can imagine manipulating the parameters in the Excel file so that the image shown with vertex 0 at top could be rotated to any other vertex in the polygon, from 1, ... , $n-1$. If the image is rotationally symmetric of order 2, 3, or 6, the number of rotations required would be divided by 2, 3, or 6. This means that there are 3, 2, and 5 distinct versions of the images in Figures 5.a, 5.b, and 5.c, respectively, before the image once more looks the same as when vertex 0 is at the top.

The issue to which we now turn is how to rotate an image by manipulating the parameters j , k , v , and w . This is most simply examined by considering an odd polygonal image ($n = 5$) with obvious starting point such as shown in Figure 8.a. The goal is to find a rule that allows us to rotate the sharp (36°) apex at 0 to each of the other vertices: 1, 2, 3, and 4. The initial starting values are $j_0 = 2$, $k_0 = 3$, $v_0 = 2$, and $w_0 = 3$. This implies that the legs of the sharp apex are $0-j$ and $0-k$ and the isosceles triangle bases are created using $v-w$.

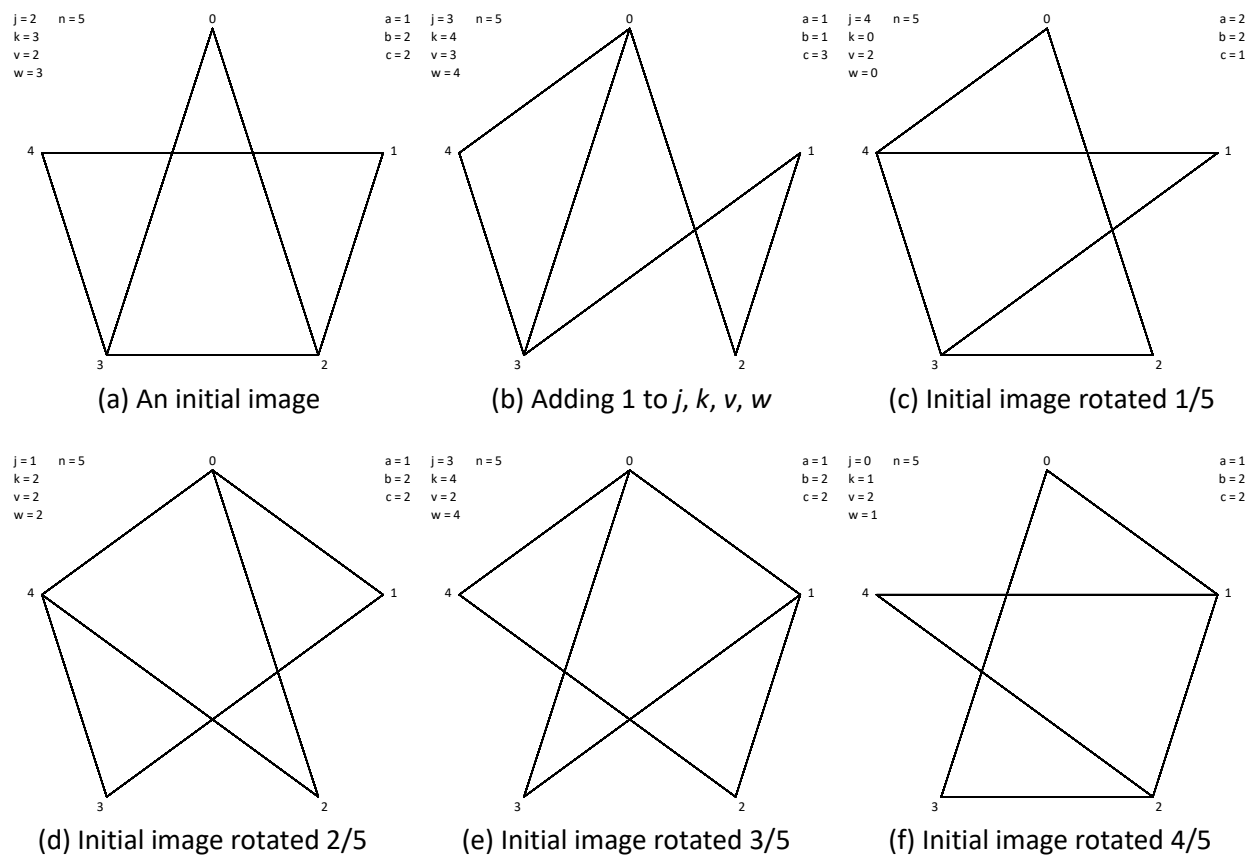


Figure 8. Rotating images clockwise around the circle

The initial guess of adding one to each of the parameters produces Figure 8.b, an entirely different image, still isosceles, but now with two 36° angles. The problem with adding 1 to all four parameters is that two of these parameters (j and k) are tethered at one end to 0 while v and w belong to a single line and thus increasing both by 1 produces different angles, and hence a different image. Therefore, the first adjustment is to increase j , k , and w by 1 while keeping v fixed. This produces Figure 8.e because the $v-w = 2-4$ baseline now has an even number of arcs between 2 and 4 so that the two-line apex vertex opposite the baseline occurs at the midpoint vertex, 3.

To get the vertex to 1, we need to add 2, not 1 to each value and we need to make one modification to the resulting numbers because sometimes that sum will be larger than $n-1$. When this happens, we want the vertex that represents the remainder once the sum is divided by n , so we use Excel's MOD function mentioned earlier (Excel states: MOD(number,divisor) returns the remainder after a number is divided by a divisor). The general rule which produced images 8.c – 8.f from 8.a is given by the following equations:

$$(1) \quad j_r = \text{MOD}(j_0 + 2 \cdot r, n), \quad k_r = \text{MOD}(k_0 + 2 \cdot r, n), \quad w_r = \text{MOD}(w_0 + 2 \cdot r, n), \quad \text{and } v_r = v_0$$

for $r = 0, 1, \dots, n-1$. The r in these equations represents the vertex to which the initial image has rotated (with $r = 0$ being the initial image). These equations are replicated on the crème-colored area at the bottom of Figure 2 which introduces the dashboard for the Excel file.

It is worth focusing once more on Figure 8.e where the two-line apex vertex is at 3. According to equation (1) this occurs when 6 is added to each initial value of j , k , and w . But we discussed above that this same image is the result of adding 1 to each of those values. Adding 6 is the same as adding 1 when you are doing modular arithmetic, base 5. Many students who have not been exposed to modular arithmetic may be confused by this. Nonetheless, everyone will instantly be able to answer the question: What time will it be 2 hours from now if it is 11 o'clock now? The 1 o'clock answer follows from the fact that 13 is 1 modulus 12. This is the same arithmetic we just did, albeit with a 5 hour clockface.

Rotating the image when n is even has the same issue noted in the move from Figure 8.a to 8.b. Consider Figure 6.b. No specific values for j_0 , k_0 , v_0 , and w_0 were noted that produced that image based on a 10-gon with 6 two-line vertices. Study that figure for a moment and determine values for these four parameters that could produce this image. Go to the Excel file and verify your results. Next, add 1 to your values of j_0 , k_0 , and w_0 and verify that you have just turned the image into Figure 6.c which notably has no two-line vertices but the same angles as in 6.b. The reason this happened is straightforward. Your initial values of j_0 , k_0 , v_0 , and w_0 had to have two-line vertices at the midpoint of both arcs on either side of a line defining that direction (because each arc has an even number of arc segments), in all three directions. Once 1 is added to one vertex endpoint for each line, then all arcs on both sides of a given line are odd.

Due to the rotational symmetry noted above for even n , there would be 5 rotational images for Figure 6.b. These are easily obtained in the Excel file by using equations (1) typed in the yellow cells for j , k , and w (shown in Figure 2) with n , j_0 , k_0 , v_0 , w_0 , and r from cell M12. The rotational values for r would be 1, 2, 3, and 4 ($r = 5$ would produce the same image as $r = 0$). Additionally, there are 5 rotational versions of 6.c which can be obtained by tricking equation (1) to give odd rotations. The trick is to use 0.5, 1.5, 2.5, 3.5, and 4.5 for r (since in equation (1), r is multiplied by 2). Alternatively, one could simply redefine initial values of j , k , and w as being one larger than the values you initially chose and then use equation (1) directly as above.

One can use this file to examine specific classes of triangles. For example, you guarantee right triangles if n is even and one of j or k is set equal to $n/2$. And you can guarantee isosceles triangles by setting $v = w = 0$ (which produces horizontal lines) and $k = n-j$ for $j = 1, 2, \dots, \text{INT}(n/2)-1$ for both even and odd n . Additionally, when n is even, a second set of isosceles images with slanted bases occur when $v = 0, w = 1$

and $k = n+2-j$ for $j = 2, 3, \dots, n/2$. The horizontal base and slanted base isosceles triangles in Figures 6.b and 6.c are examples of these styles as are the equilateral versions in Figures 9.a and 9.b for $n = 24$.

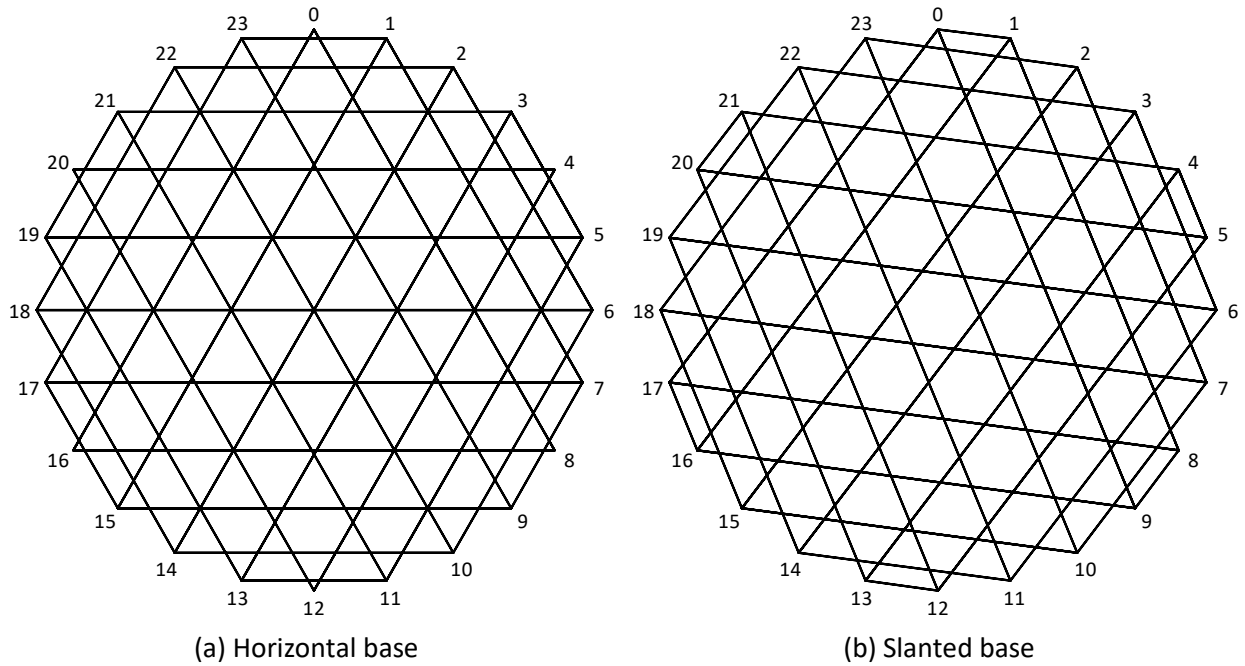


Figure 9. Two versions of equilateral triangles on 24-gon

Both images in Figure 9 exhibit 6th order rotational symmetry and have six lines of reflection. The six lines are at $j - (j+12)$ in 9.a and $j+0.5 - (j+12.5)$ in 9.b for $j = 0, 2, \dots, 10$. And, since $n = 24$, there are four rotationally distinct images for each panel because $4 = 24/6$. Setting $r = 0, 1, 2$, and 3 using equation (1) produces these images ($r = 4$ would place vertex 20 at the top instead of 0 which provides an identical image to the ones shown in Figure 9).

We have made substantial use of horizontal lines in these images because one can obtain a horizontal line regardless of n . This is simply an artifact of the way the file is constructed with vertex 0 always at the top of the circle (and polygon). Given this, 1 and $n-1$ always have the same y value so the line connecting these vertices will always be horizontal (as will j and $n-j$ for all j from 1 to $\text{INT}((n-1)/2)$). The same cannot be said of vertical lines. Given the location of vertex 0, vertical lines connecting vertices will only occur when n is even. In this instance, the sum of vertex values must be $n/2$ (from vertex 0 to the right) and $3n/2$ to the left of vertex 0.

Finally, one can examine images with parallelograms by withdrawing the third line in any of the above images. The way to do this in Excel is to type 0 in the cell for v , and $=j$ in the cell for w . The only restriction on j and k is that they cannot be within 1 of each other or a zig-zag pattern results (like Figure 1.b).