Draft discussion for File 10:

Counting Sharpest Triangles
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(The first two sections are from File 8, the final section coves the general solution.)

1. Counting sharpest triangles

When angles are created from regular polygonal vertices using two consecutive vertices for the endpoint of the legs, with vertex of the angle at another polygonal vertex, a sharpest angle has been formed. Add a third segment which touches the legs of the angle formed at two distinct points and a triangle is formed. The images created using this triangle as the basis for three sets of parallel lines connecting vertices will produce a number of similar triangles of various sizes. The triangles in the image thus created are readily counted using the sharpest angle (or apex) vertices because those vertices must, by construction, be located at the vertices of the polygon. Prior exploration (File 7) has shown that when $n = 2k+1$ is an odd regular polygon, and the sharpest apex triangle is isosceles, the count of triangles, $T(n) = ((n-1)/2)^2$ or, to say the same thing, $T(k) = k^2$. What happens when we relax these assumptions? For instance, what happens if we use even sided polygonal vertices? And, what happens if we do not require isosceles triangles? We examine these issues in turn, but before we do, it is worthwhile to digress momentarily and discuss the angles created using vertices of a regular *n*-gon.

The central angle of a regular *n*-gon is $(360/n)°$ because each of the *n* arcs of a regular *n*-gon is equal sized, and they must sum to 360°. Consider an inscribed triangle created using 3 vertices of the *n*-gon. If we count the number of vertices between each of the triangle's vertices and call these numbers *a*, *b*, and *c*, then we know that each is a whole number and they must sum to *n*. The inscribed angle theorem tells us that the angles opposite these arcs are $(180a/n)°$, $(180b/n)°$, and $(180c/n)°$. (Note that, as expected, these angles sum to 180° because $a + b + c = n$.) Put another way, the triangles created using vertices of polygons have angles that are multiples of $(180/n)°$. It also means that $(180/n)°$ is the size of the smallest angle that can be made using the vertices of a regular *n*-gon.

2. Sharpest angle right triangles in even regular polygons

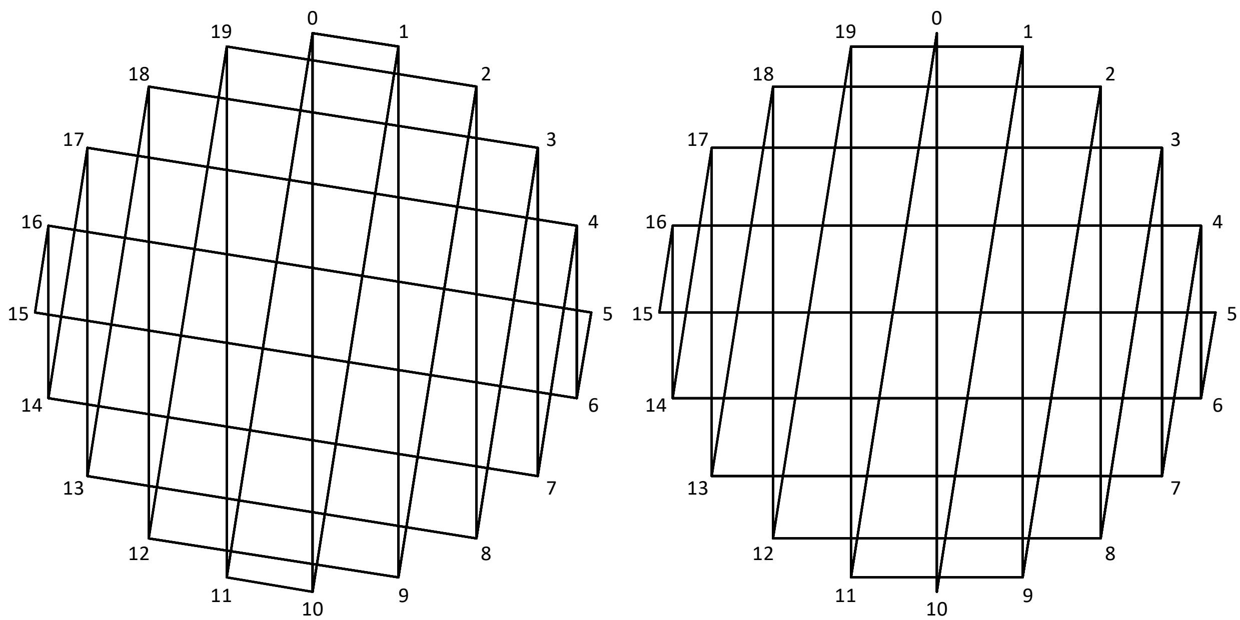
When *n* is even, one can no longer achieve an isosceles triangle if two consecutive vertices form the smallest angle of the triangle (here we use $n = 2k+2$ so that *n* represents the k^{th} even polygon, the square being the first). The closest one can get to isosceles to have 1, *k*, and $k+1$ as *a*, *b*, and *c*. The largest angle in this instance is right $(90° = (180\cdot(k+1)/(2k+2))°)$.

Figure 1 shows the two possible configurations for $n = 20$ and $k = 9$. Polygon vertices are numbered in a clockwise fashion starting with 0 at the top. Both configurations share the same lines in the two directions forming the sharpest angle; one is vertical and the other is the steepest non-vertical positive slope. The difference is the positioning of the third line which forms triangle bases. In Figure 1.a the bases are shallow negatively sloped lines and in Figure 1.b they are horizontal.

Figure 1.a has vertices that are more readily discernable since all three vertices of the largest triangle are vertices of the polygon. There are two such largest triangles, both have hypotenuse 0-10 with right angled vertices at either 1 or 11. (Line segments between vertices *i* and *j* is denoted as *i*-*j*.) All triangles

except these two have one, or at most two, vertices on the polygon. The other vertices are at the intersection of the additional lines that are parallel to the three sides of the largest triangles. Each of these triangles have right angles that are formed at the intersection of the two slanted lines.

The right angles in Figure 1.b, are however, more readily visible because two of the sets of parallel lines are vertical and horizontal. In this case, the largest hypotenuses are from 0-11 and 1-10 with right angles just beneath 0 or just above 10 at the intersections of the lines 1-19 or 9-11, with 0-10.



(a) Slanted base (b) Horizontal base

Figure 1. Two versions of sharpest apex right triangles

Two points are worth noting about the two images in Figure 1: 1) although the triangles in (a) are similar to those in (b), they are not congruent across panels; and 2) both images have the same number of triangles. Let us begin by discussing the issue of size.

Consider the side of the triangle from 15-16. In (a) that is a leg, and in (b), that is a hypotenuse. Thus, the triangle in (a) is larger. By contrast, the triangle with side from 14-16 leads to the opposite conclusion because that side is now the hypotenuse in (a) but the leg in (b). And consider the two largest triangles in both panels. Two sides are common to both triangles, the lines from 0-11 and 1-10. In (a), those lines are legs while in (b), they are hypotenuses, put another way, the largest triangles in (a) are necessarily larger than those in (b) (even for different values of n). No single conclusion can be drawn other than to know that if both lines between vertices are sides of a triangle, then the one that has that side as a leg is larger than if that side is the hypotenuse.

The easiest way to count these triangles is to count apexes from side to side. Start at the left-most vertex which is not an apex for any triangle (but is the base for a triangle). In both images in Figure 1 this is vertex 15, ¾ of the way around the circle. From here, follow the zig-zag path from apex to apex, counting as you go. In Figure 1, the path is 15 to 16 to 14 to … to 4 to 6 to 5, ¼ of the way around the circle. This path stops at all n vertices. The count of triangles increases from 0 to 9, one at a time, with the first count of 9 occurring at vertex 0, then a second 9 occurs at vertex 10 at the bottom of the 20-

gon followed by 8 at vertex 1, then … back to 0 at vertex 5. The sum of these 20 numbers can be written as:

$$T(20) = 0 + 1 + \ldots + 8 + 9 + 9 + 8 + \ldots + 1 + 0 .$$

If we remove the zeros and place the second half beneath the first half we have,

$$\begin{aligned} T(20) = {} & 1 + 2 + \ldots + 8 + 9 \\ & + 9 + 8 + \ldots + 2 + 1 . \end{aligned}$$

Instead of summing horizontally, sum vertically and we obtain,

$$T(20) = 10 + 10 + \ldots + 10 + 10 .$$

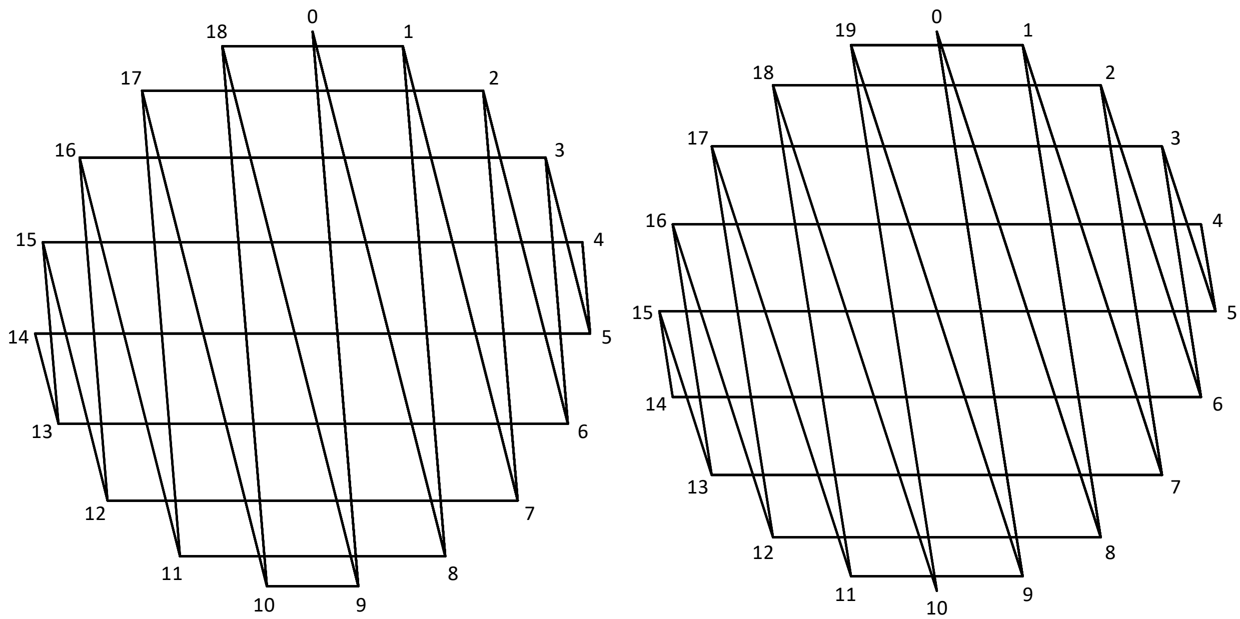
Each vertical sum is 10 and there are nine 10s. Put another way, $T(20) = 90$. This can be generalized to any even n recalling that the k^{th} even polygon has $n = 2k+2$ sides and vertices. Note that $k = (n–2)/2$. The number of sharpest angle right triangles in this n-gon is

$$\begin{aligned} T(n = 2k+2) = {} & 1 + 2 + \ldots + (k-1) + k \\ & + k + (k-1) + \ldots + 2 + 1 \\ = {} & (k+1) + (k+1) + \ldots + (k+1) + (k+1) \text{ , (} k \text{ times, or)} \end{aligned}$$

(1) $\quad T(n = 2k+2) = k\cdot(k+1)$.

3. Sharpest Apex Obtuse Triangles

We now turn to the more general situation of sharpest apex scalene triangles using polygonal vertices. The same rule applies whether n is even or odd. As noted above, the sum of angles in this situation is $(b + c)\cdot(180/n)° = 180\cdot(1 – 1/n)°$ where b and c are whole numbers that sum to n – 1. Let c be the larger of the two numbers and b be the smaller (since we are excluding isosceles triangles from this discussion). In this instance, $c = n – b – 1$ and $c > n/2$ meaning that the angle is obtuse. Figure 2 shows two examples of this for situation.



(a) n = 19, (1, 8, 10)·(180/19)°　　　　(b) n = 20, (1, 8, 11)·9°

Figure 2. Two versions of sharpest apex obtuse triangles inscribed on n-gons with horizontal base

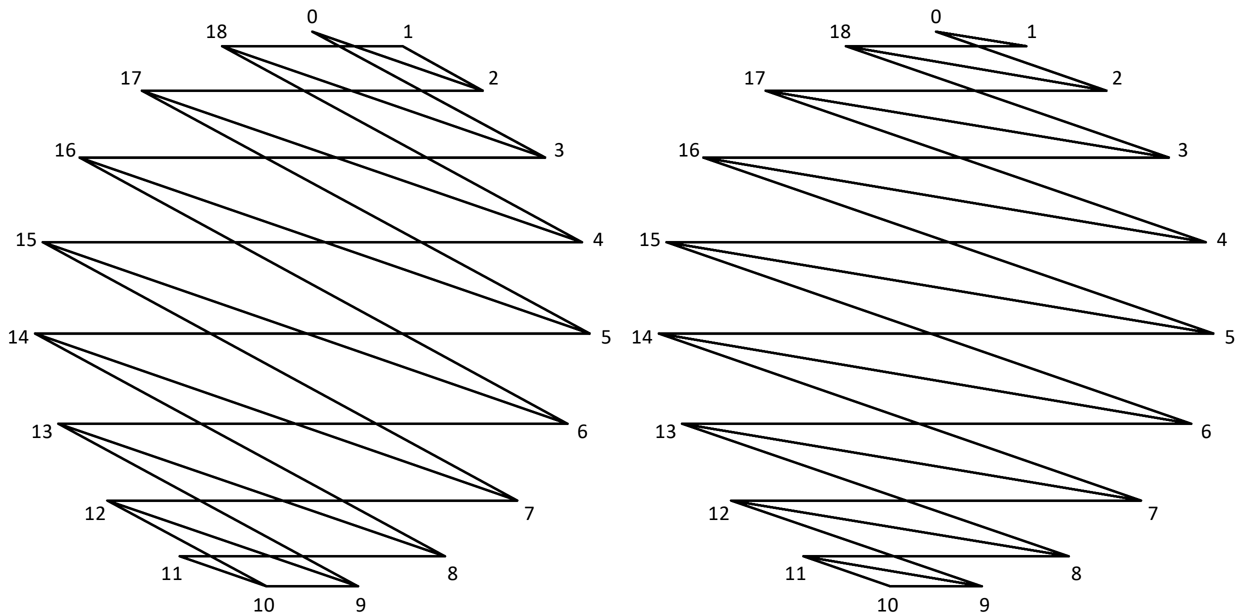
Many images are possible, and they can be explored using the Excel file that created these images (instructions for exploration are provided there). The two shown in Figure 2 are "closest-to-right-angle" solutions for an even and odd n-gon with horizontal base in order to most easily describe the counting pattern in this situation. A horizontal base is shown because, from a counting perspective, it does not matter whether the base is horizontal or slanted and the horizontal base allows easy explanation. The obtuse angle in both cases in Figure 2 are readily seen using the lowest horizontal line, starting at 9 and ending at 18 with angular vertex at 10 in (a) and 11 in (b). Rather than focus on the obtuse angle, consider the acute base angle, b.

We can describe the sharpest apex triangle by reference to size of the polygon and number of segments included in the acute base angle, b (since the obtuse angle is $c = n - b - 1$). The same zig-zag counting works in this situation except now the counting stops at b (in Figure 2, b = 8) and remains at b until returning back to zero at the other side of the image. As with Figure 1, there is an initial vertex that has no apex but does have a base (in Figure 2, vertices 4 and 14 satisfy this condition). Counts progress from 0 to 1 to … to b then eventually decline from b back to 0 on the other side of the polygon. This takes up $2\cdot(b + 1)$ vertices and the total count of these vertices uses equation (1). The other $n - 2\cdot(b + 1)$ vertices take on value b. The total triangle count for triangles created from an n-gon with is therefore given by

$$(2) \qquad T(n, b) = b\cdot(b + 1) + b\cdot(n - 2\cdot(b + 1)) .$$

The triangle count is therefore 80 in Figure 2.a and 88 in Figure 2.b.

The same pattern occurs when b is small, but the pattern is a bit harder to see initially. Figure 3 shows two 19-gons, the first with b = 2 and the second with b = 1. The vertices with apex count of zero in Figure 3.a are 1 and 11, but in Figure 3.b they are 10 and 1.



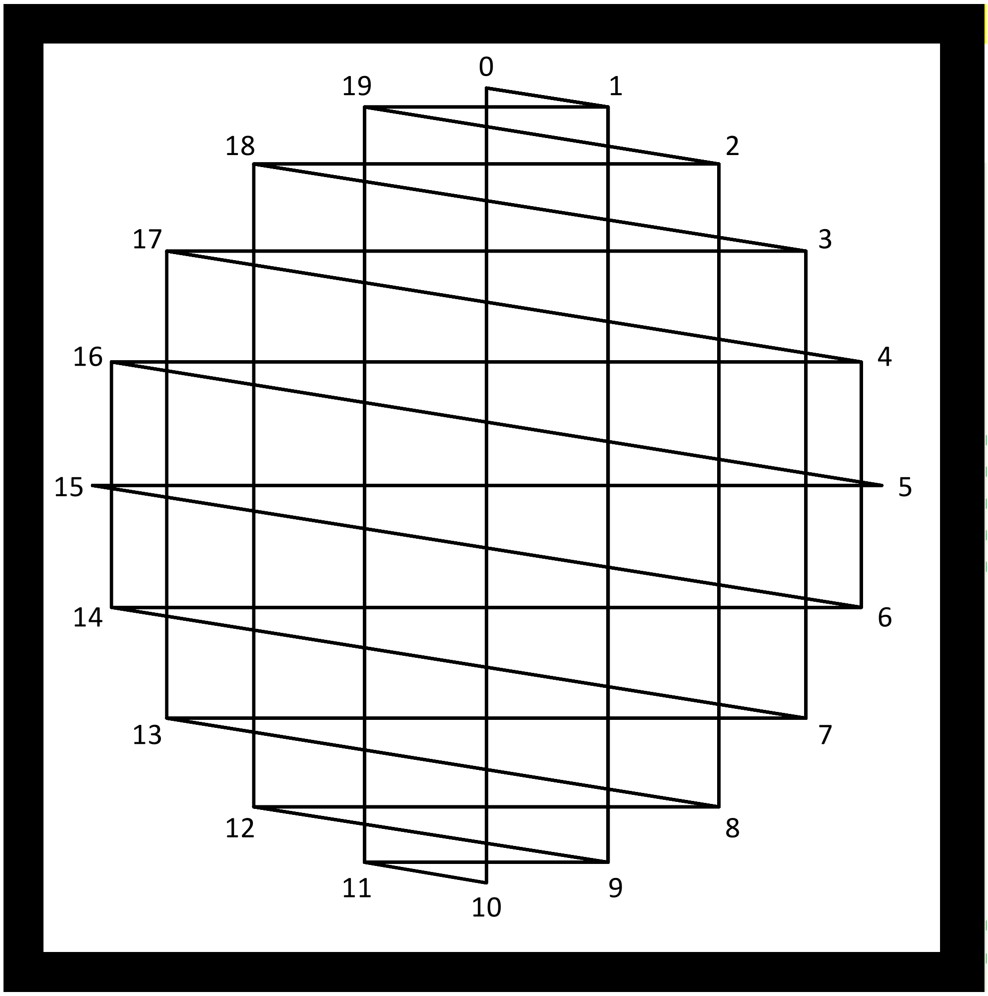
(a) triangle angles (1, 2, 16)·(180/19)°   (b) (1, 1, 17)·(180/19)°

Figure 3. Images of sharpest apex obtuse triangles using 19-gon vertices with horizontal base

Vertex 1 in Figure 3.b has the same size angle as at vertex 0 (since one of the base angles is also sharpest, making the triangles isosceles), but in order to avoid double counting, we only count triangles using the two non-horizontal legs. Put another way, the count from 10 to 11 to 9 to 12 … 2 to 0 to 1 in (b) is 0, 1, 1, …, 1, 0 where the first count of 1 (at vertex 11) is the triangle with vertices (10, 11, 9) and the last one at 0 is created from apex angle created using vertices 1-0-2. Visual inspection verifies that T(19, 2) = 32 in (a) and T(19, 1) = 17 in (b) as does equation (2).

The Excel file allows the exploration of regular *n*-gons with $3 \leq n \leq 31$. In addition to the sharpest angle images discussed here, you are free to explore images created using any sets of vertex pairs of your choosing. The dashboard for the Excel file is shown in Figure 4 with instructions for use. You should verify that the image shown is Figure 1.b rotated 90°. A worthwhile exercise would be to consider what you would need to change to create the same rotated image for Figure 1.a or how you might create a vertically reflected version of an image. One final point is worth noting: if you relax the sharpest apex assumption, internal triangle apexes will occur. This, of course, makes the counting more complex.

Create your own triangular patterns on regular polygons



1 To copy the above image to Word, open a Word document.
2 Return to Excel and click on cell B2, use down arrow to go to B3.
3 Once in B3, hold down shift key and use right arrow to move to J3 then down arrow to J28.
4 Once image is highlighted, click Copy (or CtrlC). Go to Word. Click the arrow beneath Paste.
5 Click Paste Special, Picture (Enhanced Metafile).
This will produce an image without borders.
(If you have borders, you included column K and/or row 29).
If you want to include the black border, highlight A2:L30 and follow instructions 4 and 5.

Images are created by defining three non-parallel lines between vertices.

20 ***n*** **Polygon (2 < *n* < 32)** **For simplicity the first two lines use vertex 0 = (0, 1).**

Show lines ☑ Line 1 is 0 to *j* = 1 *j* = 1, ..., *n* -1. Line 2 is 0 to *k* = 10 *k* ≠ *j* , *k* = 1, ..., *n* -1.

Show circle ☐ Line 3: 1st vertex, *v* = 0 *v* = 0, ..., *n* -1. 2nd vertex, *w* = 0 *w* ≠ *v* , *w* = 0, ..., *n* -1.

Show labels ☑ *Note:* The third line need not include *j* or *k* , although that is fine.

You can write in the green area. You can manually enter numbers in the yellow cells, or you can insert equations in those cells.
The yellow cells have been labeled so you can refer to them by name. For example entering, =int(n/3) in Q3, =n-j in V3,
1 in R4, and =n-v in W4, produces "near equilateral" isosceles triangles. These are exact when n is divisible by 3.
To create parallellograms, set v=0 and w=j.

Figure 4. Images from the Excel file used to create triangular images on polygonal vertices