Discovering Number Patterns in Triangle Counting

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Don Chakerian
Professor Emeritus, University of California Davis

and

Stephen E. Erfle
Professor, Dickinson College

Abstract

Students learn mathematics by deeply investigating number patterns and patterns embodied in geometric figures. Regular polygons provide a frame for using parallel lines to create similar shapes, the simplest of which is the triangle. Configurations of such triangles provide a natural opportunity to connect number patterns to geometric patterns.

Discovering Number Patterns in Triangle Counting

§1. Introduction

Mathematics has been characterized as the science of patterns. Students studying mathematics in their early years are encouraged to investigate number patterns and patterns embodied in geometric figures. Appropriately guided investigations dealing with counting are the basis for what Professor Sherman Stein has dubbed *The Triex: Explore, Extract, Explain,* where one explores and gathers data in a particular problem area, extracting some sort of order or pattern, ultimately seeking an explanation of what has been found. This is what most curriculum guides wisely recommend for teaching introductory mathematics in a way consistent with the nature of mathematics and how it is actually practiced.

The main purpose of this article is to provide some raw material for such investigations and projects with a class of nonstandard, but what we consider interesting, geometric configurations where counting the number of similar triangles leads to formulas involving familiar number patterns that are in themselves of interest. These geometric configurations can also serve as an opportunity for students across the curriculum to consider elementary properties of triangles, circles, and regular polygons.

Before presenting our examples and suggested investigations in sections 3 and 4, we consider some basic facts about the number patterns that arise in analyzing these figures. The accompanying Excel file could be used by teachers to show students completed images or to create materials that students could use to create their own constructions based on regular polygons with up to 30 sides.

§2. Some Number Patterns and Formulas.

Most of us have heard the popular anecdote about the ten-year-old Carl Friedrich Gauss in class being asked to calculate the sum of the integers from 1 to 100. The young prodigy proceeded to reverse the order of the numbers, so corresponding terms of the original and the reversed set added to 101. Since the sum of all 100 of these 101's gives twice the required sum, he noted that 50 copies of 101 would give the result he wanted. Thus, he was able to almost instantly compute the required answer, $50 \cdot 101 = 5050$, in his head.

Of course, this trick generalizes to the sum of the first n positive integers. If we denote $S = 1 + 2 + \ldots + n$, then also $S = n + (n-1) + \ldots + 2 + 1$, so

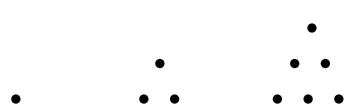
$$\begin{aligned} 2S = S + S &= (1 + n) + (2 + (n-1)) + (3 + (n-2)) + \ldots + (n + 1) \\ &= (n + 1) + (n + 1) + (n + 1) + \ldots + (n + 1) \quad (n \text{ terms}) \\ &= n \cdot (n + 1), \end{aligned}$$

which gives us the formula

$$(1) \qquad 1 + 2 + \ldots + n = n \cdot (n + 1)/2.$$

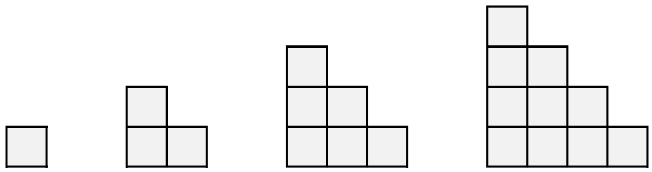
This clever method also provides a way to derive a formula for the sum of the terms of any arithmetic progression. It is worth noting that this trick was known to Archimedes around 200 BC (Stein 1999).

The relation (1) is often presented as an easy example of a formula to be verified by mathematical induction. But in early grades one would like students to experiment with the sums 1, 1 + 2, 1 + 2 + 3, …, and make conjectures that might lead to (1). In other words, this is an opportunity for applying "Triple-X." The sum $1 + 2 + \ldots + n$ is the n^{th} *triangular number*, which we shall denote by Δ_n, so $\Delta_n = 1 + 2 + \ldots + n = n \cdot (n + 1)/2$, and is so named because it can be represented geometrically by a triangle of dots:

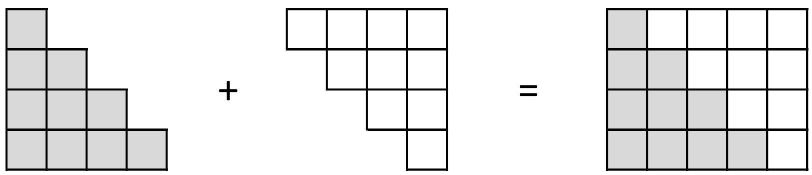


$\Delta_1 = 1$ $\Delta_2 = 3$ $\Delta_3 = 6$ …

One often appeals to a geometric representation of a slightly different sort in justifying (1). We can represent Δ_n as a triangle of unit squares, as below:



With this representation, we can place two copies of Δ_n together to form a $n \times (n+1)$ rectangle (here is a picture for $n = 4$):



Then consideration of areas gives $2\Delta_n = n \cdot (n + 1)$. One can see this to be a geometric representation of little Gauss's inspiration.

Another related example, which will be of use later in this article, is that the sum of the first n positive *odd* integers is n^2. That is,

(2) $1 + 3 + 5 + \ldots + (2n-1) = n^2$.

This is another case of a formula that can be offered as an example to be verified by mathematical induction. But, here again, in early grades we would want students to experiment with the sums 1, 1 + 3, 1 + 3 + 5, … and come up with (2) as a conjecture, and eventually get into a discussion of why (2) is true in general.

One can justify (2) in the same way little Gauss obtained the result in (1). Namely, we denote the sum by $S = 1 + 3 + \ldots + (2n-1)$ and reverse the order as $S = (2n-1) + (2n-3) + \ldots + 3 + 1$. Adding the equations leads to

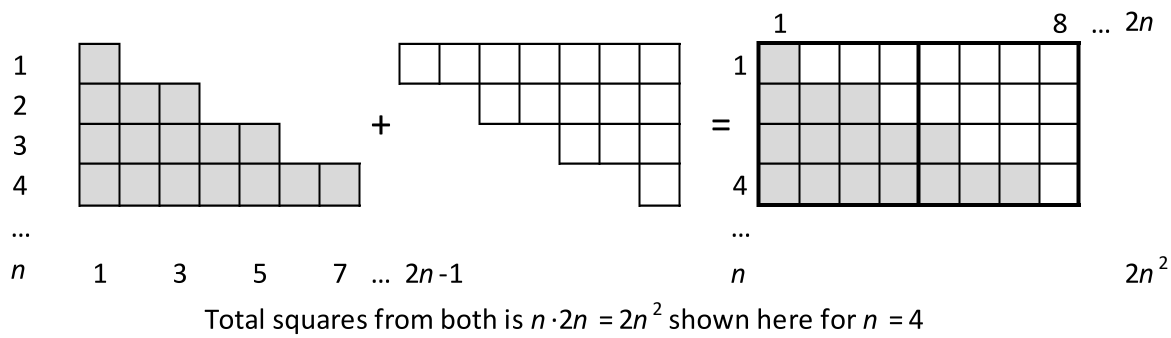
$$2S = (1 + (2n-1)) + (3 + (2n-3)) + \ldots + ((2n-1) + 1)$$

$$= 2n + 2n + \ldots + 2n \quad (n \text{ terms})$$

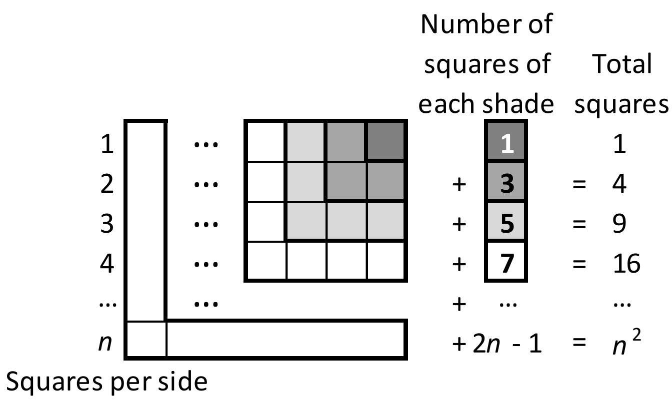
$= n \cdot (2n) = 2n^2$.

From this we get (2).

We can also use a geometric representation analogous to what we did for Δ_n:



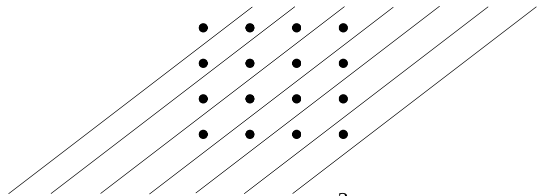
On the other hand, the traditional geometric representation of (2) involves noticing that an $n \times n$ square grid of unit squares can be partitioned into L-shaped pieces (called gnomons), each of which contains an odd number of unit squares:



Here is a final preparatory number pattern that is of later use. Observe that $1 + 2 + 1 = 2^2$, $1 + 2 + 3 + 2 + 1 = 3^2$, $1 + 2 + 3 + 4 + 3 + 2 + 1 = 4^2$, … . This suggests that for any positive integer n,

(3) $\quad 1 + 2 + \ldots + (n-1) + n + (n-1) + \ldots + 2 + 1 = n^2$.

For obvious reasons, this can be called "up the hill and down again." The reader will have little trouble proving the formula by breaking the sum in (3) into two pieces and applying (1) to each piece. But a very nice geometric representation is given by using an $n \times n$ square array of dots and counting the dots in a diagonal fashion, illustrated below in case $n = 4$:



$1 + 2 + 3 + 4 + 3 + 2 + 1 = 4^2$

There are numerous geometric counting problems that lead to triangular numbers, square numbers, sums of the sort we see in (1) and (2), and more complex number patterns (Conway and Guy 1995). These are useful to teachers in assigning student projects and investigations and in enriching traditional courses. For example, counting the total number of squares of all sizes in an $n \times n$ square grid leads to a count of $1^2 + 2^2 + \ldots + n^2$. For instance, a 3×3 grid of unit squares contains $14 = 1^2 + 2^2 + 3^2$ squares of various sizes. Aside from finding this expression, there then might arise the question of expressing this sum of squares in a simple and concise form. It turns out that

(4) $\quad 1^2 + 2^2 + \ldots + n^2 = n\cdot(n + 1)\cdot(2n + 1)/6.$

This formula can be proved with a straightforward application of mathematical induction, but we would like a verification similar to our previous examples. There are many ways to do this and we would like to give a particularly elegant proof of the relation for the reader. However, doing this at the moment will divert us from the course we want to follow at this point, so we present the argument in an Appendix at the end of the article. The proof puts many of the ideas we have discussed to good use.

§3. Isosceles Triangles in Regular Polygons.

The configurations we will consider are generated by sets of parallel chords connecting vertices of a regular polygon. For our purposes, it will suffice to consider a regular n-gon P, where $n \geq 3$, inscribed in the unit circle centered at (0, 0) in the (x, y) plane. It will also be convenient to assume P has a vertex at the point (0, 1); this is the "top vertex" of P.

The isosceles triangles, to be counted in our later counting problems, are created by drawing three sets of parallel chords connecting vertices of P. For the moment we postpone describing how to do this in general; it will be helpful to get started with the special case $n = 12$. So, we let P be a regular 12-gon (regular dodecagon) inscribed in the unit circle, with a top vertex at (0, 1). To clearly describe the construction, we label the vertices of P with the numbers 0, 1, 2, …, 11, so that the vertices different from (0, 1) receive the usual designations of hours on a standard clockface, with (0, 1) being assigned 0 rather than 12 (the informed reader might detect a whiff of the faint odor of clock arithmetic here).

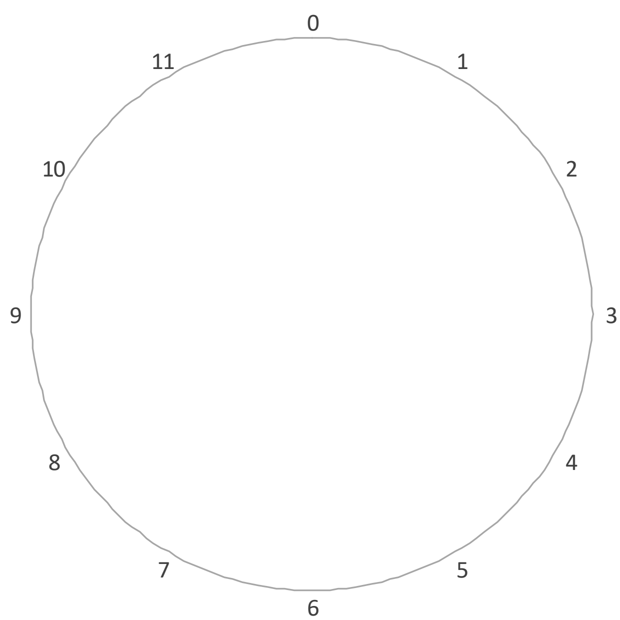


Figure 1. Enumerating vertices given $n = 12$

Now we proceed to draw the three sets of parallel chords that yield a collection of similar isosceles triangles of the sort we want. These triangles will have their bases on horizontal chords connecting pairs of vertices of P. If we want the apex angle (angle opposite the base) to be as small as possible, it turns out that in this special case of $n = 12$ our triangles will have to be similar to the isosceles triangle with vertices at 0, 5, and 7, so the apex angle will be 30° = $\pi/6$ radians. The three families of parallel chords are pictured in Figure 2 below, forming a configuration containing many isosceles triangles similar to the 0, 5, 7 triangle. We have

(a) horizontal chords parallel to the chord from 5 to 7,

(b) chords of steep positive slope parallel to the chord from 0 to 7,

(c) chords of steep negative slope parallel to the chord from 0 to 5.

It is evident that the isosceles triangles we see in the figure are all similar to the 0, 5, 7 triangle, since corresponding sides are parallel (so corresponding angles are equal).

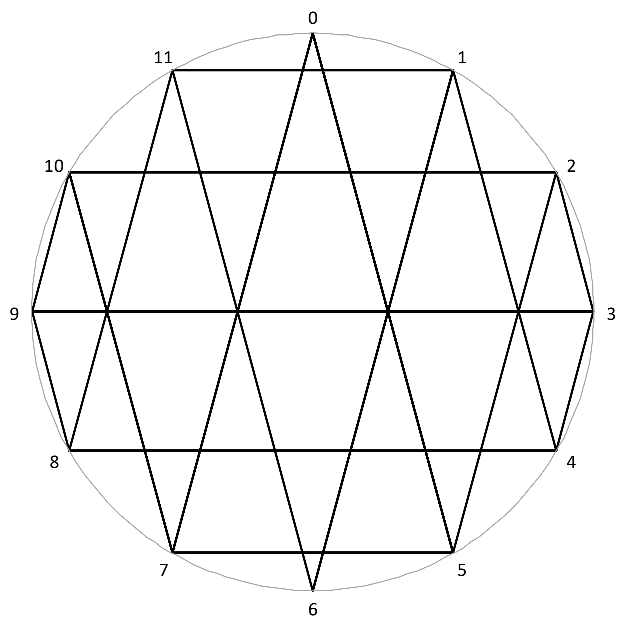


Figure 2. Sharpest apex isosceles triangles with horizontal bases for $n = 12$

Figure 3 below shows another configuration generated in an analogous way, containing isosceles triangles with 30° apex angles, but *not* horizontal bases, and actually containing a total number of such triangles *different* from that in Figure 2. Every other configuration of this nature turns out to be one of these two.

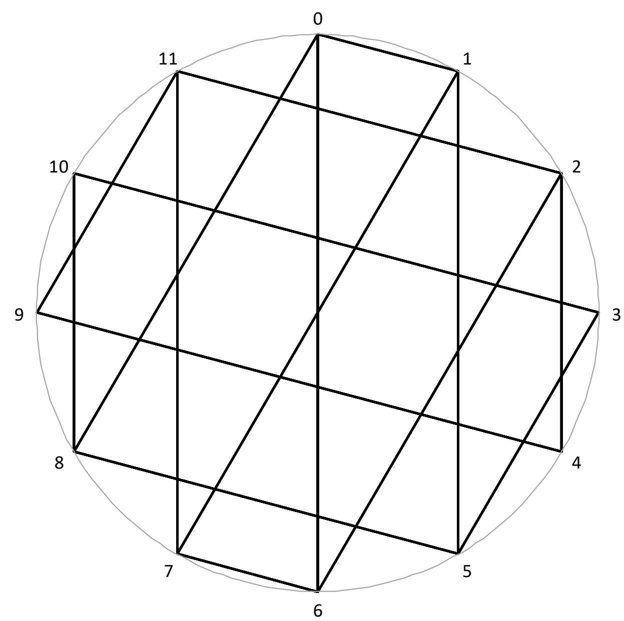


Figure 3. Sharpest apex isosceles triangles with slanted bases for n = 12

At this point, the reader may want to verify that the configuration in Figure 2 contains 38 isosceles triangle of the type we are constructing, while the configuration in Figure 3 contains 42 such triangles (Beware: some of these triangles are "upside down").

We turn now to the general case for the construction when $n \geq 3$. While the general construction is carried out much like the case $n = 12$, there are some variations depending on whether n is even or odd. We continue to inscribe our regular n-gon P in the unit circle, with one vertex at (0, 1). We label the vertices 0, 1, 2, …, n-1, in clockwise order, starting with the label 0 for the top vertex (0, 1). The simplest case is when n is odd.

If *n* is odd, with $n = 2k+1$, then there is no single lowest vertex, and the "lowest" vertices of P are labeled k and k+1. The triangle with vertices 0, k, k+1 has apex angle $\pi/n = \pi/(2k+1)$, the smallest possible value, and is isosceles with horizontal base given by the chord joining k to k+1 as shown in Figure 4. (Note that by the Inscribed Angle Theorem, the apex angle is half the central angle determined by the arc of the circle from k to k+1, and that central angle is $2\pi/n = 2\pi/(2k+1)$.)

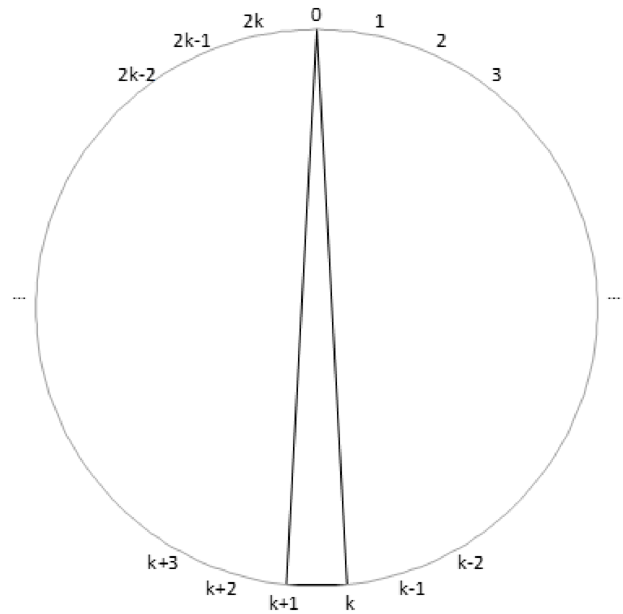


Figure 4. Partial configuration showing largest triangle when n is odd

As in the case n = 12, we now construct three families of parallel chords determined by the vertices by drawing:

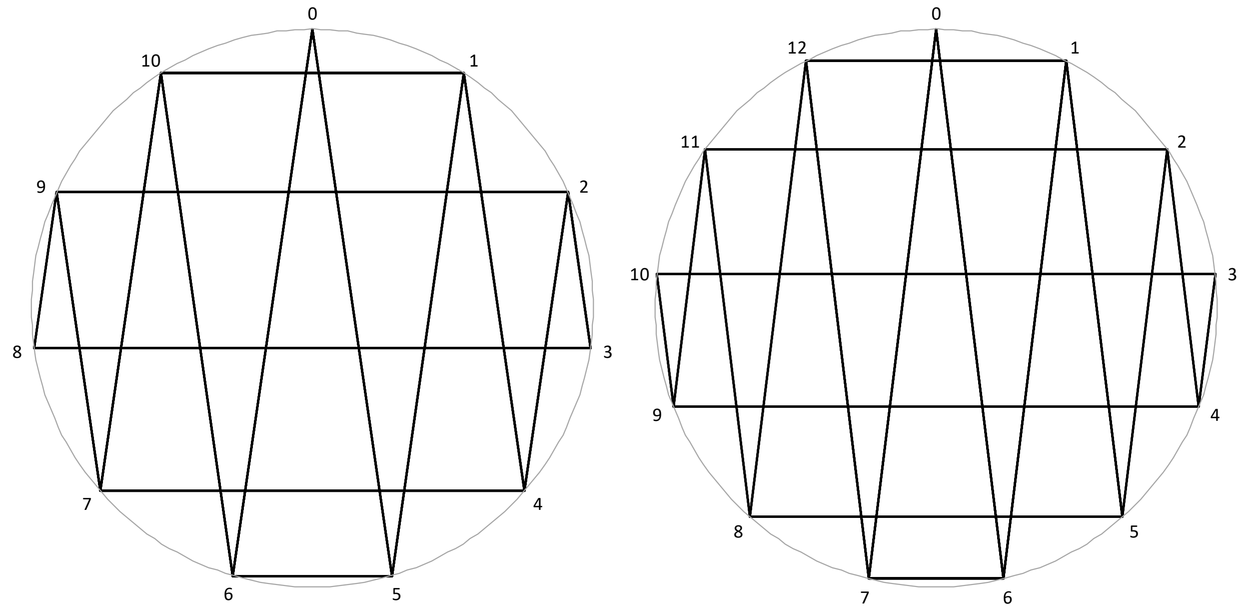
(a) horizontal chords parallel to the chord joining k to k+1,

(b) chords of steep positive slope parallel to the chord joining 0 to k+1,

(c) chords of steep negative slope parallel to the chord joining 0 to k.

Again, as in the case n = 12, we see that the isosceles triangles contained in this configuration are all similar to the triangle 0, k, k+1. Figure 5 illustrates the cases n = 11 and n = 13 (n = 2k+1 with k = 5 and k = 6). In the configuration for n odd, n = 2k+1, the horizontal chords are obtained by joining j to n-j for j = 1, 2, …, k = (n-1)/2.

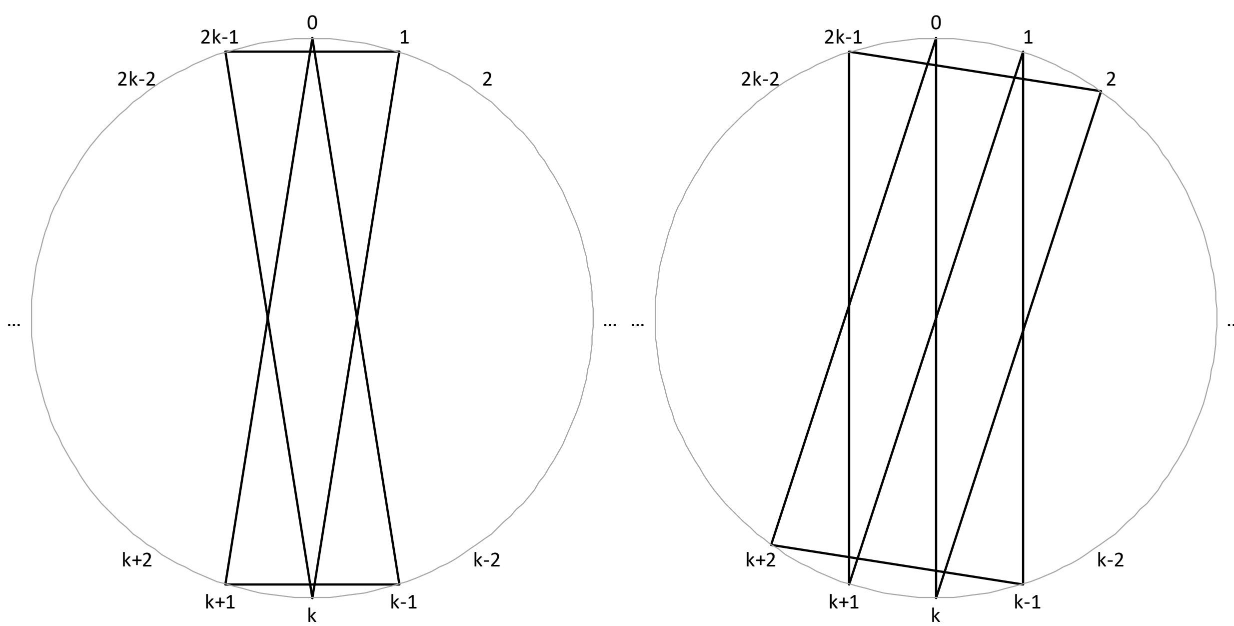
Also notice that every one of the isosceles triangles generated this way has a vertex of the n-gon as its vertex. It turns out that this gives an easy way to count the total number of isosceles triangles, as we shall see in section 4.



(a) $n = 11$ (b) $n = 13$

Figure 5. Triangles created when n is odd

Now we consider the case when ***n* is even**, with $n = 2k$. We have seen two configurations for creating triangles with apex angle 30° = π/6 radians in Figures 2 and 3 for $n = 12$. In Figures 6(a) and 6(b) we indicate how such configurations are constructed in general.



(a) Horizontal Base (b) Slanted Base

Figure 6. Partial configurations showing largest triangles when n is even

With Figure 6(a) as a basis, we draw three families of chords joining vertices, parallel to the horizontal chord joining k-1 to k+1, then parallel to the chord from 0 to k-1, and finally parallel to the chord from 0 to k+1. In Figure 6(b) we draw those parallel to the chord from k to k+1, then those parallel to the vertical chord from 0 to k, and finally those parallel to the chord from 0 to k+2. The apex angle of the triangles in both instances are of size $2\pi/n = \pi/k$ radians (the corresponding central angle determined by the arc of the circle from k-1 to k+1, or from k to k+2, has size $2\cdot(2\pi/n) = 4\pi/n$ radians, and the apex angle is half this size). Figure 7(a), (b), (c), and (d) show completed versions of both configurations when $n = 8$ and $n = 10$. We shall see that when $n = 2k$, the total number of triangles in each configuration depends on whether k is even or odd.

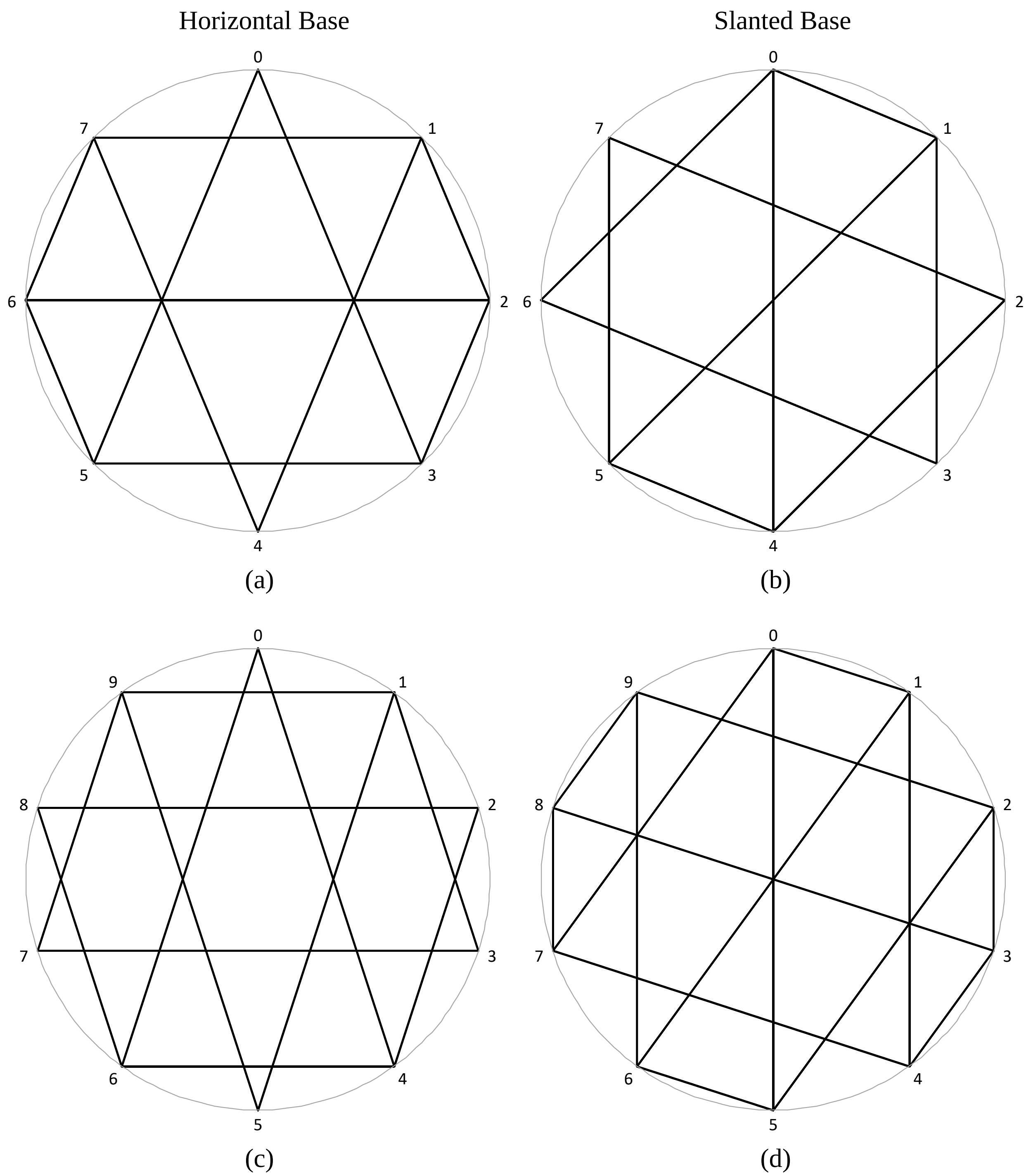
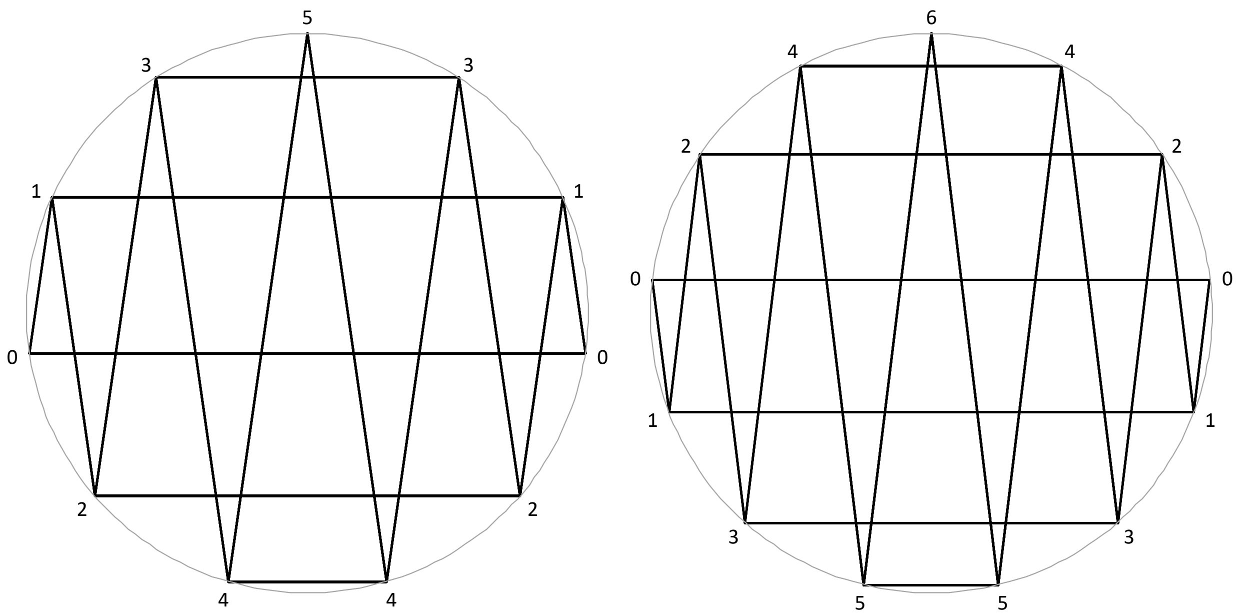


Figure 7. Triangles created when n is even: Top row, n = 8; bottom row, n = 10

§4. Counting Triangles

We now consider possible strategies for counting the total number of triangles in the configurations we have generated. The simplest case is when ***n* is odd**, $n = 2k+1$, since every one of the resulting triangles has its apex on the circle. We illustrate this in Figure 8 for $n = 11$ and 13, where we have relabeled vertices from Figure 5 so that numbers at each vertex denote the number of triangles having their apex at that point:



(a) $n = 11$, $k = 5$ (b) $n = 13$, $k = 6$

Figure 8. Counting triangles when $n = 2k+1$ is odd

The key observation here is that in Figure 8(a) we see an up and down zig-zag of steep chords with successive corners of the zig-zag labeled 0, 1, 2, 3, 4, 5, 4, 3, 2, 1, 0, where each number indicates how many triangles have apex at that corner. Thus, summing these numbers gives the total of T triangles, which, by the up the hill and down again equation (3), is $5^2 = 25$. In the same manner, from Figure 8(b) we get a total of $T = 6^2 = 36$.

Trying a few more cases will convince you that when $n = 2k+1$, you always get an up and down zig-zag of steep chords with labels at successive corners that follow the pattern:

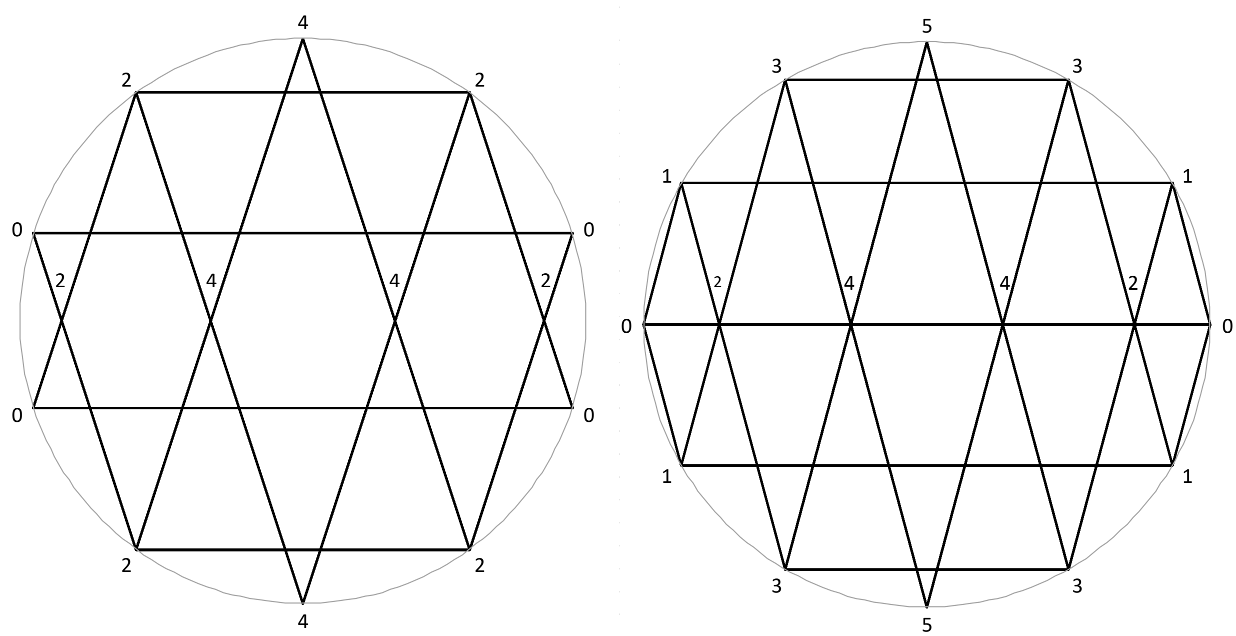
0, 1, 2, …, (k-1), k, (k-1), …, 2, 1, 0.

Notice that there are k horizontal lines in this instance. The very top vertex will be the apex of k triangles because each of the k horizontal lines in the image forms a base for one of those similar triangles, and so will label the top vertex k. From this, using equation (3) with k replacing n, we see the total number of triangles T is:

(5) $$T = 1 + 2 + \ldots + (k\text{-}1) + k + (k\text{-}1) + \ldots + 2 + 1 = k^2 = ((n\text{-}1)/2)^2,$$

when n is odd, and $n = 2k+1$. (See the later equation (7) for a more descriptive notation to express this result.)

The counting process is more complex when ***n* is even**, since we have two configurations which we have described as having horizontal bases and slanted bases. In addition, we have points on the interior to the circle that serve as triangle apexes. These interior apexes are on a diameter of the circle. In Figure 9 we have taken the horizontal bases case for $n = 10$ (Figure 7(c)) and $n = 12$ (Figure 2) and labeled points by the number of triangles with apex at that point.



(a) $n = 10$ ($n = 4k+2$ with $k = 2$) (b) $n = 12$ ($n = 4k$ with $k = 3$)

Figure 9. Counting triangles with horizontal bases when n is even

In Figure 9(a) we see 0, 2, 4, 2, 0 appearing *twice* on the circle, once on the upper arc and again on the lower arc (or alternatively, you can see *two* up and down zig-zags with corners on the circle and the labels 0, 2, 4, 2, 0 at the successive corners of each). Additionally, we have 2, 4, 4, 2 at the four interior apex vertices on the horizontal diameter of the circle. This gives a total T of triangles:

$$T = 2\cdot(2 + 4 + 2) + 2\cdot(2 + 4) = 4\cdot(1 + 2 + 1) + 4\cdot(1 + 2) = 4\cdot 2^2 + 4\cdot 3 = 28.$$

This case, $n = 10$, is an instance of $n = 4k+2$, with $k = 2$.

You may now draw the analogous configuration for the case $n = 14$ (which is $n = 4k+2$ with $k = 3$) and label each vertex of the 14-gon by the number of triangles with apex at that point, and similarly for the six interior points having triangles with apex at that point, and find the pattern 0, 2, 4, 6, 4, 2, 0 on the upper arc of the circle and the same pattern on the lower arc. Additionally, you will find the pattern 2, 4, 6, 6, 4, 2 on the interior points. Thus, the total T of triangles will be:

$$T = 2\cdot(2 + 4 + 6 + 4 + 2) + 2\cdot(2 + 4 + 6) = 4\cdot(1 + 2 + 3 + 2 + 1) + 4\cdot(1 + 2 + 3) = 4\cdot 3^2 + 4\cdot 6 = 60.$$

A few more examples will show you the pattern that emerges, giving the total number T of triangles when $n = 4k+2$ as:

$$T = 4\cdot(1 + 2 +\ldots+ (k\text{-}1) + k + (k\text{-}1) +\ldots+ 2 + 1) + 4\cdot(1 + 2 +\ldots+ k) = 4\cdot k^2 + 2k\cdot(k+1) = 6k^2 + 2k,$$

where we have used formulas (1) and (3) from section 2. To indicate that this is the total number of triangles when $n = 4k+2$ in the configuration giving horizontal bases, we introduce the function T_H, so

(6) $\quad T_H(4k+2) = 6k^2 + 2k.$

Let us note that with this notation, the earlier result for odd n in (5) becomes,

(7) $\quad T_H(2k+1) = k^2.$

What about $T_H(12)$? Figure 9(b) shows us that $T_H(12) = 2\cdot(1 + 3 + 5 + 3 + 1) + 2\cdot(2 + 4)$. In anticipation of the general case, we deal with the first term by writing

$$1 + 3 + 5 + 3 + 1 = (1 + 3 + 5) + (1 + 3) = 3^2 + 2^2,$$

thinking of formula (2) from section 2. Thus, $T_H(12) = 2\cdot(3^2 + 2^2) + 4\cdot(1 + 2) = 38$. Now 12 is an instance of $n = 4k$ with $k = 3$, so with the aim of discovering $T_H(4k)$ in general, we next look at the configuration for $n = 16$ ($n = 4k$ with $k = 4$). There you will find the pattern

$$T_H(16) = 2\cdot(1 + 3 + 5 + 7 + 5 + 3 + 1) + 2\cdot(2 + 4 + 6) = 2\cdot(4^2 + 3^2) + 4\cdot(1 + 2 + 3) = 74.$$

A few more experiments will lead you to the emerging pattern

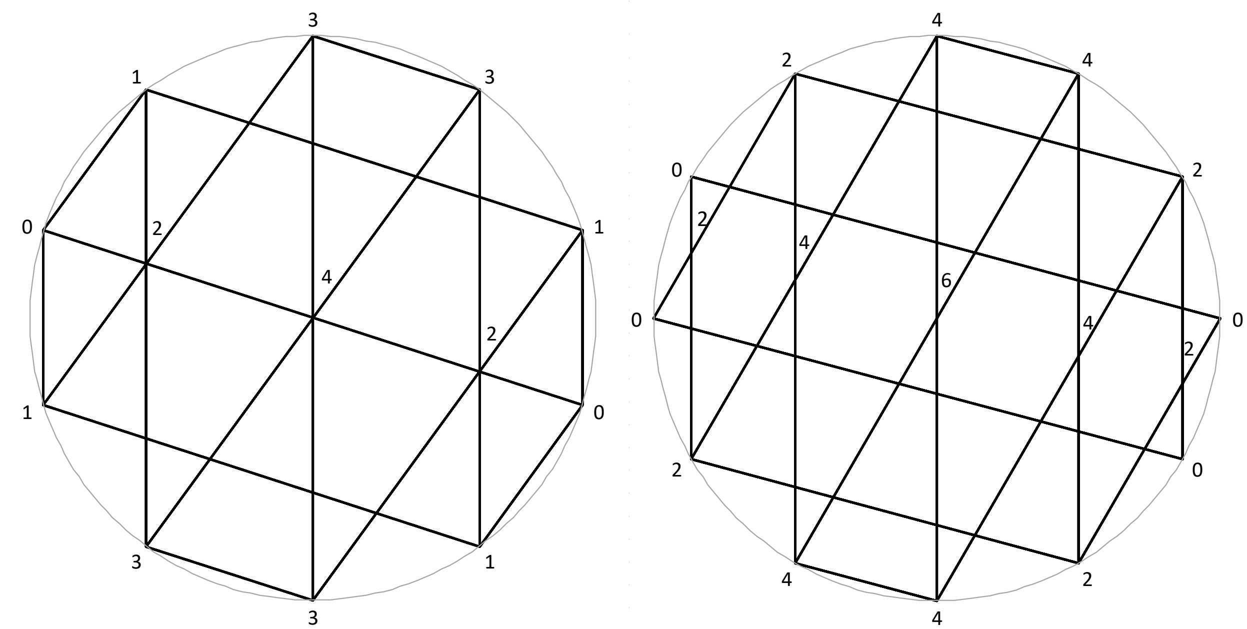
$$T_H(4k) = 2\cdot(1+3+\ldots+(2k-1)+\ldots+3+1) + 2\cdot(2+4+\ldots+(2k-2)) = 2\cdot(k^2 + (k-1)^2) + 4\cdot(1+2+\ldots+(k-1))$$

$$= 2\cdot(2\cdot k^2 - 2\cdot k + 1) + 2k\cdot(k-1),$$

which on simplification gives us

$$(8) \qquad T_H(4k) = 6k^2 – 6k + 2.$$

Thus far we have the results for the configuration of triangles with horizontal bases. We now want to deal with the case of $n = 4k$ and $n = 4k+2$ for triangles with slanted bases. For this we use the notation $T_S(4k)$ and $T_S(4k+2)$ for the totals of such triangles. Figure 10 depicts the configurations for $n = 10$ (Figure 7(d)) and $n = 12$ (Figure 3) while labeling points by the number of triangles with apexes at that point.



(a) n = 10 (n = 4k+2 with k = 2)   (b) n = 12 (n = 4k with k = 3)

Figure 10. Counting triangles with slanted bases when n is even

One way to carry out the count in Figure 10(a) is to notice an up and down zig-zag with corners on the upper arc of the circle *and on a diameter* of the circle that has at successive corners 0, 1, 2, 3, 4, 3, 2, 1, 0 while on the lower arc of the circle we have 1, 3, 3, 1. These two patterns count every apex of a triangle in the configuration. Thus,

$T_S(10) = (1 + 2 + 3 + 4 + 3 + 2 + 1) + 2(1 + 3) = 4^2 + 2 \cdot 4 = 24.$

With further examples you will see the pattern for $n = 4k+2$:

$T_S(4k+2) = (1 + 2 + \ldots + 2k + \ldots + 2 + 1) + 2(1 + 3 + 5 + \ldots + (2k-1)) = (2k)^2 + 2k^2.$

Thus, we find

$$(9) \qquad T_S(4k+2) = 6k^2.$$

It is not hard to see that there are other ways to use the labels to arrive at the same result.

We are finally confronted with those configurations of triangles with slanted bases when $n = 4k$; that is, with finding $T_S(4k)$. In Figure 10(b) we have such a case with n = 12. A way to proceed with the count here is to notice that 0, 2, 4 appears *four* times on the circle, while 2, 4, 6, 4, 2 appears in the interior. Thus

$T_S(12) = 4\cdot(2 + 4) + (2 + 4 + 6 + 4 + 2) = 8\cdot(1 + 2) + 2\cdot(1 + 2 + 3 + 2 + 1) = 8\cdot3 + 2\cdot3^2 = 42.$

If you next draw the configuration for $n = 16$ ($n = 4k$ with $k = 4$) and label the relevant points, you will find analogously,

$T_S(16) = 4\cdot(2+4+6) + (2+4+6+8+6+4+2) = 8\cdot(1+2+3) + (1+2+3+4+3+2+1) = 8\cdot6 + 2\cdot4^2 = 80.$

With a few more examples there emerges the pattern that gives:

$$T_S(4k) = 4\cdot(2 + 4 +\ldots+ 2(k\text{-}1)) + (2 + 4 +\ldots+ 2k +\ldots+ 4 + 2)$$

$$=8\cdot(1 + 2 +\ldots+ (k\text{-}1)) + 2\cdot(1 + 2 +\ldots+ k +\ldots+ 2 + 1) = 4k\cdot(k\text{-}1) + 2k^2.$$

Regrouping we have

(10) $T_S(4k) = 6k^2 – 4k.$

Students confronted with the challenge of making these counts are likely to find many ingenious alternative methods. For example, it is possible (while somewhat more difficult) to count the bases of the relevant triangles.

Our emphasis here has been to present material for exploration, in the spirit of the “X-plore” and “X-tract” parts of Stein’s “Triple-X.” In order to fulfill the “X-plain” part we should actually prove the validity of the formulas (6), (7), (8), (9), and (10). Giving more or less rigorous arguments for these would require a digression taking us beyond the main intent of this article. However, it might be helpful to briefly describe how this might be done. What we suggest could also be viewed as an approach to counting the triangles in a recursive manner.

As a first example, consider the simple case where n is odd. The crux of the matter is to see that if n is odd and you compare the configuration for an n-gon to that for a $(n+2)$-gon (i.e., going from $n = 2k+1$ to $n+2 = 2(k+1)+1$), the labels for the $(n+2)$-gon configuration are obtained by adding 1 to each label in the *n*-gon configuration, together with two 0's. Thus, the count of triangles *increases by n*. So, starting with $n = 3$, where $T_H(3)=1$, we get

$T_H(3)=1$, $T_H(5)=1 + 3$, $T_H(7)=1 + 3 + 5$, $T_H(9)=1 + 3 + 5 + 7$, …,

so, we have $T_H(2k+1) = 1 + 3 + 5 +\ldots+ (2k-1) = k^2$, which gives us formula (7).

One can see these additional *n* triangles by imagining how Figure 8(b) might have been created from Figure 8(a). The apex angle gets tighter (from π/11 to π/13 radians) in order to create room for two new vertices equally spaced on the circle. With the peak vertex fixed, those vertices above the horizontal diameter in Figure 8(a) (with triangle counts 1 and 3) move closer to the peak and those below the horizontal diameter (with triangle counts 0, 2, and 4) move closer to the bottom with both movements making space for two new vertices to be added (at vertices 3 and 10 using the labeling in Figure 5(b)). The horizontal line drawn between these two "new" vertices creates 11 new triangles, one for each of the adjusted vertices from Figure 8(a) but none at the two newly created vertices that, in this instance, are just above the horizontal diameter. This is why each of the 11 vertices in the adjusted image from Figure 8(a) increase by one while the two new vertices have no triangles for which that vertex is the apex.

Similar ideas work for the cases where *n* is even. For example, when comparing the configuration of horizontally based triangle for $n = 4k$ with that for $n = 4(k+1) = 4k+4$, it is possible to see that the labels on the circle increase by $8k$ in total sum while the labels on the interior increase by $4k$ in total sum. Thus, the total number of triangles increases by $12k$, so we have: $T_H(4k+4) = T_H(4k) + 12k$. Therefore, starting with $k = 1$, we get

$T_H(4) = 2$, $T_H(8) = 2 + 12$, $T_H(12) = 2 + 12 + 24$, $T_H(16) = 2 + 12 + 24 + 36$, …,

and in general,

$T_H(4k) = 2 + 12\cdot(1 + 2 + 3 +\ldots+ (k-1)) = 2 + 6k(k-1) = 6k^2 - 6k + 2$, giving us formula (8).

The reader may want to check the other cases. For instance, you will find that in going from the configuration of horizontal bases with $n = 4k+2$ to that with $n = 4(k+1)+2 = 4k + 6$, the increase in triangles is $12k + 8$. So, starting with $k = 1$ we have

$T_H(6)=8$, $T_H(10) = 8 + (12\cdot 1 +8)$, $T_H(14) = 8 + (12\cdot 1 + 8) + (12\cdot 2 + 8)$, … , and in general

$T_H(4k+2) = 8\cdot k + 12\cdot(1 + 2 + 3 +\ldots+ (k-1)) = 8k + 6k(k-1) = 6k^2 + 2k$, confirming formula (6).

Finally, it is worth noting that we can see geometrically how $T(4k)$ and $T(4k+2)$ relate to one another. The easiest case is $T_S(4k+2)$ versus $T_S(4k)$ because $T_S(4k)$ is $4k$ smaller than $T_S(4k+2)$ according to formulas (9) and (10). Using the strategy noted above for odd polygons and starting with Figure 10(b) where $n = 12$, we consider making each apex angle a bit smaller (from $\pi/6$ to $\pi/7$ radians) to accommodate two new vertices, located between the pairs of vertices labeled 0 in Figure 10(b). These two newly created points are connected by the slanted diameter line which passes through all interior apex vertices thereby creating no new interior apex triangles but increasing by one the number of triangles whose apexes are on the circle, except for the two newly created vertices which have value 0. This is an increase of $4k$ triangles. A similar geometric argument relates to the horizontal case depicted Figures 9(a) with 9(b), but care must be taken in this instance because $k = 2$ in 9(a) and $k = 3$ in 9(b). Nonetheless, note that by decreasing the apex angle (from $\pi/5$ to $\pi/6$ radians) and drawing in the horizontal diameter connecting the two new vertices, we increase all apex counts by one while none of the interior apex counts change because the horizontal diameter line has gone through each interior apex. There are 10 more triangles in Figure 9(b) than 9(a), one at each of the vertices that formed 9(a), albeit moved a bit closer together.

§5. Conclusion and Extensions

This paper suggested strategies for counting triangle in configurations created by three families of parallel lines joining vertices of a regular n-gon. We chose to restrict ourselves to a specific class of those lines which produce sharpest apex isosceles triangles, and we showed that the triangles in such configurations can be counted using a single formula when n is odd, and by one of four formulas when n is even.

The accompanying Excel file, *TrianglesOnRegularPolygons.xlsx*, has two sheets. The main worksheet, *Triangles from Lines,* allows users to create configurations from the vertices of regular polygons with 3 to 31 sides using five parameters: n, and four user defined vertices that set up three non-parallel lines. This allows the user to recreate the sharpest apex isosceles images in this paper, and to also examine alternative constructions. Images can be shown with or without the circle on which the vertices are located as well as vertex labels. These images can be exported to Word using the instructions provided on the sheet.

The general case of triangles on regular polygons can be categorized by the angles generated under the various constructions. The sum of angles in a triangle is π radians and the angles of a triangle that can be constructed from vertices of a regular n-gon must be of the form: $a\pi/n$, $b\pi/n$, and $c\pi/n$ radians, where a, b, and c are positive integers that sum to n.

The *Polygon Vertices* sheet allows users to create vertex point image worksheets for student use with pencil and ruler on paper. Two worksheet formats are suggested but others are readily created as the need arises.

Appendix

Here we give a proof of formula (4) for the sum of squares of the first n positive integers. What we present here is a variation of an extraordinary argument found in *The Book of Numbers*, a book containing a dazzling array of colorful representations of interesting and remarkable number patterns (Conway and Guy 1995). But, like formula (1), this formula was known to Archimedes (Stein 1999).

We base the proof on a triangle of odd integers of the following sort (the case $n = 4$ of the first n odd integers is depicted below):

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 3 & & \\ & 1 & & 3 & & 5 & \\ 1 & & 3 & & 5 & & 7 \end{array}$$

The sum of the entries in the triangle is $1^2 + 2^2 + 3^2 + 4^2$ because of formula (2), so summing the entries in the triangle in three different ways gives $3\cdot(1^2 + 2^2 + 3^2 + 4^2)$. We shall see that this sum is also $9\cdot(1 + 2 + 3 + 4) = 9\Delta_4$, giving us $3\cdot(1^2 + 2^2 + 3^2 + 4^2) = 9\Delta_4$. But this is the case $n = 4$ of

(A) $\quad 3\cdot(1^2 + 2^2 + \dots + n^2) = (2n + 1)\cdot\Delta_n$,

which, on multiplying both sides of equation (4) by 3, is seen to be equivalent to the result (4) that we want to prove.

So, our aim is to obtain (A). We illustrate how to do this in the case $n = 4$. Begin by observing that, by (2), we have

(B) $\quad 1^2 + 2^2 + 3^2 + 4^2 = 1 + (1 + 3) + (1 + 3 + 5) + (1 + 3 + 5 + 7)$.

We rewrite this (note the resemblance to Gauss's trick):

(C) $\quad 1^2 + 2^2 + 3^2 + 4^2 = 1 + (3 + 1) + (5 + 3 + 1) + (7 + 5 + 3 + 1)$.

Finally, notice that the entries in the triangle consist of one 7, two 5's, three 3's, and four 1's, so

(D) $\quad 1^2 + 2^2 + 3^2 + 4^2 = 7 + (5 + 5) + (3 + 3 + 3) + (1 + 1 + 1 + 1)$.

Consider summing the left-hand and right-hand sides of (B), (C), and (D) separately. The left-hand sides sum to

(E) $3\cdot(1^2 + 2^2 + 3^2 + 4^2)$.

Next, we add the right-hand sides of (B), (C), (D) by <u>adding terms vertically, column by column</u>. This gives

(F) $$9 + (9 + 9) + (9 + 9 + 9) + (9 + 9 + 9 + 9).$$

(This miracle will seem less mysterious if you see what happens when you sum the right-hand sides of only (B) and (C) in the same vertical manner.)

But (F) is

(G) $9\cdot(1 + 2 + 3 + 4) = 9\Delta_4$.

Since (E) and (F) are equal, we have

(H) $3\cdot(1^2 + 2^2 + 3^2 + 4^2) = 9\Delta_4$,

which is the result (A) we want in case of $n = 4$.

If you go through this exercise for some larger value of n, it will become evident that it works in the same way for any value of n, with the Δ_4 copies of 9 in (F) replaced by Δ_n copies of $(2n+1)$. In the general case we are writing $1^2 + 2^2 + \ldots + n^2$ in three different ways that lead, on summing these equations, to

$$3\cdot(1^2 + 2^2 + \ldots + n^2) = (2n + 1)\cdot(1 + 2 + \ldots + n) = (2n + 1)\cdot\Delta_n,$$

just as requested in (A).

References

Conway, John H., and Richard Guy. 1995. *The Book of Numbers*. Corrected Edition. New York, NY: Copernicus.

Stein, Sherman. 1999. *Archimedes: What Did He Do Beside Cry Eureka?* 1st Edition. Washington, DC: The Mathematical Association of America.