Alternative Visions of Perfect Squares:
An Exercise that ties Geometric to Numeric Patterns

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Abstract

Once students know multiplication, they tend to have a firmer grasp on perfect squares than other multiples. This paper keys off that knowledge and shows multiple geometric and numeric patterns in what comprises a perfect square. The accompanying Excel file is a self-contained teaching tool for in-person and remote classrooms. The file also lends itself to independent exploration by those interested in connecting geometric and numeric patterns. Once a young learner sees the connection between these patterns, they can challenge their older siblings (or parents) to guess how many triangles they see in a given image, knowing full-well that the answer is quickly within their grasp. If anything can encourage a youngster to want to learn, it is being able to pull one over on their older siblings and parents!

For use with: PerfectSquares.xlsx

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One of the most enjoyable parts of teaching mathematics is observing students make connections between one area of mathematics and another. This exercise uses an Excel file that builds a geometric pattern using the n odd vertices of regular polygons, $3 \leq n \leq 31$. The resulting images include isosceles triangles of various sizes and the goal of the exercise is to count the triangles. Systematic counting strategies are discussed, and the pattern of those counts is examined in both geometric and numeric venues. This exercise is aligned with the fourth Common Core Standard of Mathematical Practice, SMP.4, *Model with mathematics* and SMP.7, *Look for and make use of structure* [1]. (References to grade level Standards of Mathematical Content are denoted SMC.) This material, while not traditionally part of elementary education materials, conforms with a number of the SMCs for Grades 3 through 5 but it would be of interest in later grades as well. As such, it would be of interest to students of varying grade levels involved in after-school activities such as math circles and accelerated learning programs.

The Excel file has two main sheets, *Sharpest Triangles* and *Square*; the first takes a geometric perspective and the second a numeric perspective. Start on the *Sharpest Triangles* sheet and start with an image with a fairly large number of vertices such as the $21 = n = 2k + 1$, with $k = 10$) version in Figure 1. Then ask the students, "Does anyone have an idea how many triangles of various sizes are shown in this image? Once we are done with this exercise, you will be able to answer this very easily." Figure 1 shows the basic structure of the sheet. Click-boxes at the top control the image and click-boxes down the side provide lecture prompts that allow you to explain the image from scratch as you talk the class through the exercise. These prompts are organized into the five topic areas shown in Table 1. This sheet was built to help encourage in-class discussion, whether that discussion is in-person or remote.

Figure 1. Partial image showing basic structure of the *Sharpest Triangles* sheet

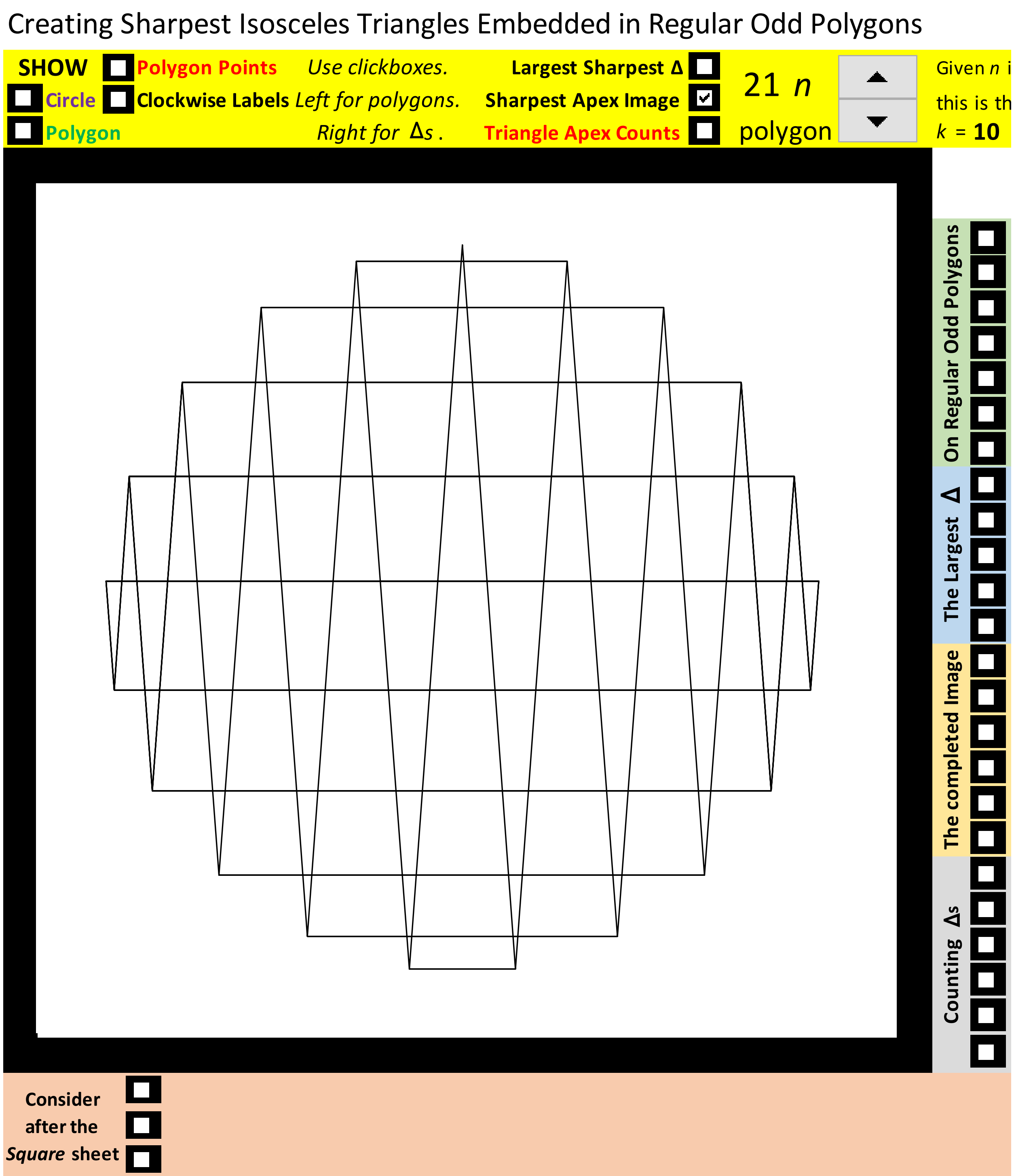


Table 1. Clickable *Sharpest Triangles* discussion points and questions in five topic areas

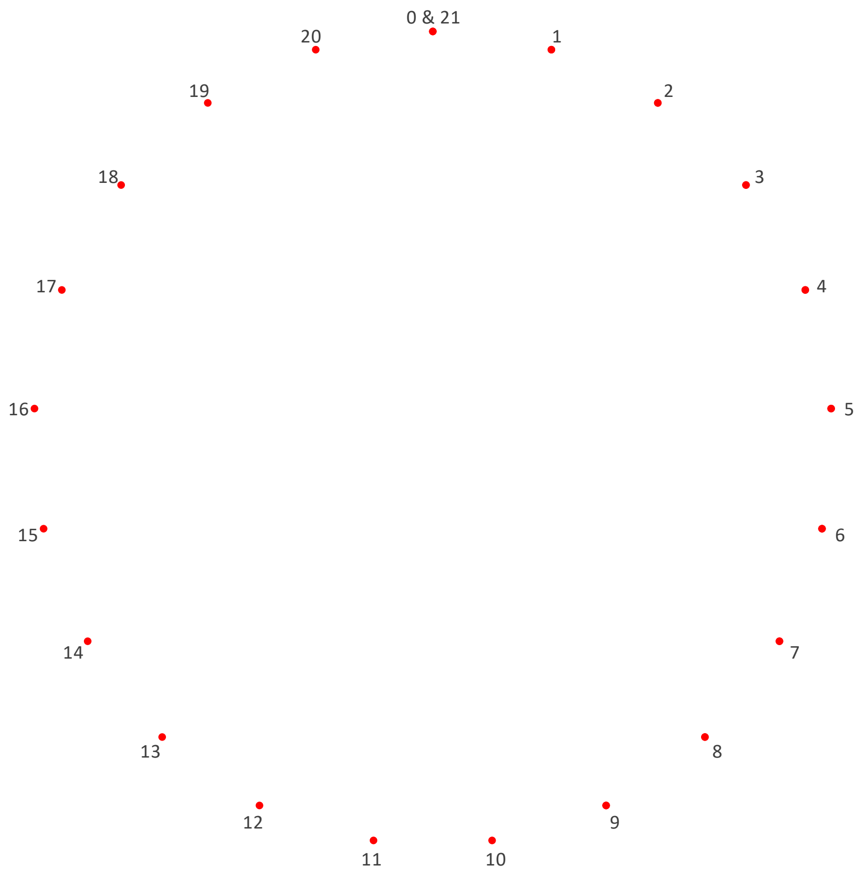
On Regular Odd Polygons	1	In this figure, each regular odd polygon has a fixed vertex at the top of the circle (click A4 on and I2-I4 off, and use the scroll arrows).
	2	Label this top point both 0 and n, and other points just like a clock that has n hours rather than 12 (click C3 on).
	3	This means that the bottom of the polygon will be flat and there is vertical symmetry.
	4	This symmetry means that there are k paired vertices at the same height in the (x, y) plane.
	5	In this instance, all vertices from 1 to k have a horizontal counterpart in vertices n-1 to n-k = k+1 (click C2 and C3 on and A4 off).
	6	Put another way, we could draw k horizontal line segments between pairs of vertices.
	7	The same thing is true in other directions except that then the k parallel lines are no longer horizontal.
The Largest Δ	1	Three non-parallel lines can be used to create triangular images.
	2	We are interested in looking at vertices which create the sharpest apex angle using these vertices.
	3	We are also going to restrict ourselves to isosceles triangles (click W2 on to see apex and base angles).
	4	Consider the largest such sharpest angle isosceles triangle having a horizontal base and slanted legs.
	5	It will have vertices 0, k, and k+1 -- show this triangle by clicking I2 on and I3 off (this triangle shares a common base with the polygon).
The completed Image	1	To create the final image, we connect all other vertex pairs which have lines parallel to each of these three lines.
	2	To do this, we must draw the other k-1 parallel lines in these three directions. Once done, the completed image emerges (click I3 on).
	3	Scroll n from 3 to 31 and watch how the images develop (click Circle, Polygon, and Polygon Points off for sharpest image).
	4	How would you describe what happens each time n increases?
	5	The apex angle gets a bit sharper and a new "fold" or "wave" happens with the largest horizontal line just above or below the middle.
	6	Notice that the new fold is downward pointing when k is odd and upward pointing when k is even.
Counting Δs	1	We wish to count all triangles of various sizes in this image. Call this number of triangles $T(n)$.
	2	Move from 3 to 5 to 7 and see what happens (remember, some triangles are "upside down") then think about larger n.
	3	There are various ways to do this, but the easiest is to use apex vertices (since all apex vertices are also the polygon's vertices).
	4	Therefore, for each polygon vertex we can attach a count of triangles with apex at that point (click I4
	5	Notice that as we move toward the side from the top or bottom, apex counts decline by 2 per vertex … to see why, focus on bases.
	6	There are various ways to sum apex counts around the circle, go to the **Square** sheet to see one elegant method.
After Square Sheet	1	If n is large, and you decide to add numbers of apex counts starting at the top and going clockwise, it will soon become tedious.
	2	Instead, start at one of the two vertices with apex counts of 0 located at the end of the wave and follow the zig-zag pattern from one side to the other. Notice the number pattern. From here, $T(n)$ should be clear.
	3	It turns out that there is another interesting way to visualize k^2 using something called gnomons. To read about that, go back to the **Square** sheet and click the box in AA4. Finally, click Q6 for two additional methods.

Table 1 lays out the discussion points and questions which teachers could use when talking through the image on the *Sharpest Triangles* sheet. Here, we focus on how teachers could use this sheet in different grades to help achieve SMC goals for that grade.

This exercise makes heavy use of SMC.3.OA.9, *Identify arithmetic patterns*, and SMC.3.G, *Reason with shapes and their attributes*. One could discuss fractions using angles in the isosceles triangles (where students discover that: $1/n + k/n + k/n = (2k + 1)/n = n/n = 1$) using SMC.3.NF, *Develop understanding of fractions as numbers*. The main point of the exercise coincides with, SMC.4.OA.5, *Generate and analyze patterns* and SMC.4.G, *Draw and identify lines and angles, and classify shapes by properties of their lines and angles*, especially with regard to lines of symmetry and parallel lines. The triangles created from polygonal vertices are readily linked to angles and thus relate to SMC.4.MD.5 *Geometric measurement: understand concepts of angle and measure angles*. An explicit goal in SMC.5.OA.3, *Analyze patterns and relationships*, is to link patterns using two different rules, which, as noted above, is the main point of the exercise. Finally, the images analyzed are based on graphing as requested by SMC.5.G, *Graph points on the coordinate plane to solve real-world and mathematical problems.*

If you would like to have your students create their own image(s) using pencil, ruler, and paper, you can create worksheets with one or more polygonal vertex images by deciding what value(s) of *n* you wish to use as well as whether you want to provide labels. Suppose you want to provide students a 21-vertex image with labels. Set *n* to 21, click C2 and C3 on, and follow the instructions provided at N1:AA1 (reproduced at the bottom of Figure 2). The resulting image is shown in Figure 2. Such worksheets could have between 1 and 6 images per page by appropriate resizing and setting margins to minimum values.

Figure 2. Hypothetical worksheet image of 21-gon vertices and labels with transfer instructions



To copy image, click upper left corner of cell B6, hold Shift down, Arrow right > to K6 then down ∨ to row 29. Once highlighted, type Ctrl+C. Go to Word, click ∨ beneath Paste, click Paste Special, click Picture (Enhanced Metafile) and resize as necessary.

Once you have worked through the first three topic areas in Table 1, you can talk about strategies for counting the triangles in the resulting image. Because sharpest angles have been chosen, there are no interior intersection points with sharpest apex angles. All interior intersections involve base angles where the horizontal base line intersects with either positively sloped OR negatively sloped leg lines. This means that we can count all triangles by focusing attention on the apexes only which are all at polygon vertices. (You could point out that the alternate strategy of counting triangle bases requires considering triangles on both sides of each horizontal line (except the line from *k* to *k*+1).)

There are a couple of interesting patterns in the triangle apex counts (the red numbers at vertices once I4 is clicked on). First, triangle apex counts decline by two per vertex in moving toward the side from either the top or bottom. Second, the top half and bottom half alternate between all even or all odd numbers, depending on *n*. Third, all vertices except the left- and right-most vertices are apexes for at least one triangle but the left- and right-most vertices do not support an apex (but they are vertices of a triangle base). Summing triangle apex counts in order around the circle can be done, but it is, at the very least, tedious. Turn to the *Square* sheet for help.

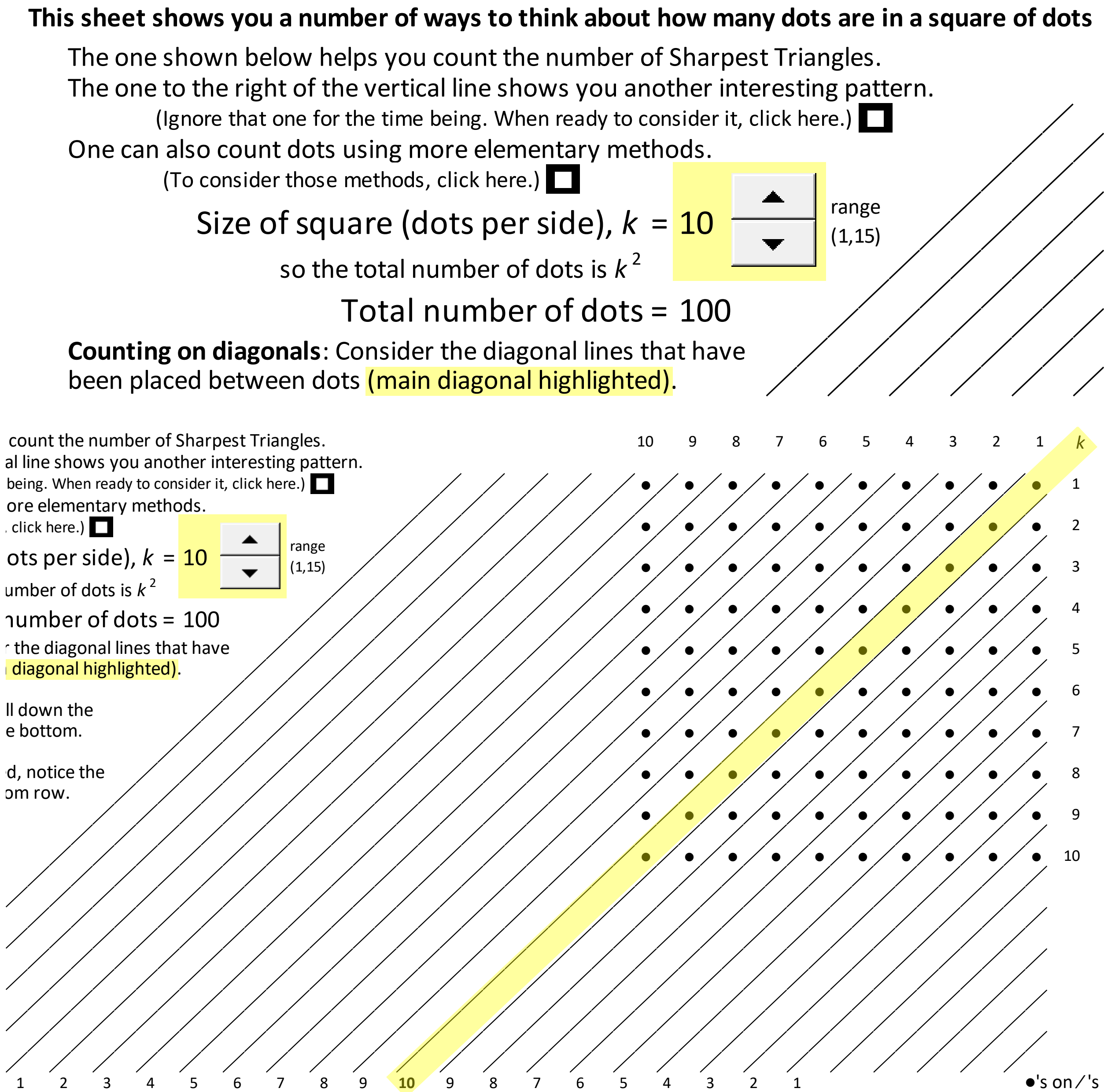
The *Square* sheet shows a square array of dots with diagonal lines between dots. Two partial images from this sheet are shown as Figure 3. The diagonals do not affect the number of dots in the square but do allow a different way to view those dots. Instead of counting by row or column (with *k* per row (or column) in each of *k* rows (or columns)), *count by diagonal*. When you do this, you see that dot counts increase from 1 to *k* then decrease back to 1. This is true for any *k* (the *Square* sheet allows $1 \leq k \leq 15$, but clearly the pattern continues for larger values of *k*). This leads to the formula

(1) $1 + 2 + \ldots + k\text{-}1 + k + k\text{-}1 + \ldots + 2 + 1 = k^2$.

This formula might well be called “up the hill and back down again.”

Return to the *Sharpest Triangles* sheet and look at apex counts one more time (turn I3 and I4 on). Instead of counting clockwise, or by quadrant, or by top and bottom, *count in a zig-zag manner starting at one of the two values of 0*. Vertex counts increase from 0 to $k = (n - 1)/2$ and back to 0 once you reach the other side. But equation (1) tells us that this must sum to k^2. The exercise is, at this point, done.

Figure 3. Two pieces of the *Square* sheet given k = 10 showing topics and diagonal elements



If you want, you can explore additional patterns involving perfect squares. Return to the *Square* sheet and click AA4 on to see an explanation of the *gnomon* pattern, an example of which is shown as Figure 4. As you increase k using the scroll bar you readily see that the square increases by adding an L-shaped odd-numbered addition called a gnomon. In particular, the k^{th} gnomon is the k^{th} odd number. Thus, another way to conceptualize k^2 is as the sum of the first *k* odd numbers

(2) $1 + 3 + \ldots + (2k\text{-}1) = k^2$.

(It is worth noting the minus sign in (2). We want the first odd number to be 1 not 3. By contrast, a triangle is the first odd polygon, so we used *n* = 2*k*+1 to tie *n* to *k* for odd polygons.)

Figure 4. Partial image from *Square* sheet given *k* = 10 discussing gnomons

k	**Now, ignore the diagonal lines. Instead, focus on the ∟ shaped dots that are added to the left and on the bottom as k increases.**	
	gnomon	The nomenclature for this "L" shaped addition is gnomon.
1	1	To count the gnomon of dots, note that adding a row adds k dots, and adding a column adds k dots, and one of those dots,
2	3	the bottom left corner, is common to both row and column. Each gnomon is therefore of the form 2k-1, where 2k-1
3	5	is the kth odd number. This means that: The sequence of gnomons is the sequence of odd numbers.
4	7	In other words, the sum of the
5	9	first k odd numbers is k^2.
6	11	
7	13	
8	15	
9	17	
10	19	
	100	= sum of the first 10 odd numbers.

The final method uses formula (1) to show the sum of integers from 1 to *k*. The clickable discussion at the bottom of the *Square* sheet, replicated in Table 2, leads to the formula for the sum of the first *k* integers:

(3) $1 + 2 + \ldots + k = k \cdot (k + 1)/2$.

Table 2. Clickable notes at the bottom of the *Square* sheet showing another use for the hill formula: Gauss addition

1	The hill formula provides a 'side door' to an even more famous pattern in numbers formula: Suppose you are asked to sum the numbers from 1 to 100?
2	If we increase *k* to 100, the hill pattern would have that sum plus the sum from 99 to 1. If we add 100, we have twice the sum from 1 to 100. Therefore, $100^2 + 100$ = twice the sum from 1 to 100.
3	Dividing by two we have: The sum from 1 to 100 = $(100^2 + 100)/2 = 100 \cdot (100+1)/2 = 5{,}050$.
4	More generally, $1 + 2 + \ldots + k = k \cdot (k + 1)/2$. This is an example of Gauss addition (click next box to learn more).
	The highlighted material may be too difficult for Grades 3 and 4.
5	The classic story goes that Carl Friedrich Gauss (1777 - 1955) recognized a pattern as a young child when asked to sum the numbers from 1 to 100.
6	He noticed that if you take a second copy of those numbers and reverse their order and put them on top of one another, something magical occurs. To see this click the next box.
7	1 + 2 + 3 + ... + 98 + 99 + 100 100 + 99 + 98 + ... + 3 + 2 + 1 Instead of adding horizontally, add vertically: 101 + 101 + 101 + ... + 101 + 101 + 101
8	Each vertical sum is the same and the top row shows how many 101s are present. Therefore, twice the sum of 1 to 100 is $100 \cdot 101$ so the sum of 1 to 100 = $100 \cdot 101/2 = 5{,}050$.
9	It is worth noting that it is standard practice to go in the opposite direction and derive the hill formula (1) from the sum of the first *k* numbers formula (3) rather than deriving (3) from (1).

As a way to wrap up the discussion, scroll up to $n = 31$ and ask: Can you figure out a way to see that the total number of triangles in the image is 225 without using formula (1)? Here is a hint: Treat each quadrant on its own and apply formulas (2) and (3). Click on cell X2 on the *Sharpest Triangle* sheet to see that answer.

REFERENCE

[1] National Governors Association Center for Best Practices (NGA Center) and Council of Chief State School Officers (CCSSO). (2010). Common Core State Standards for Mathematics. NGA Center and CCSSO. https://www.nctm.org/ccssm/