Angles of Polygons and Stars, Take 2: Two questions

The formula for the ***n***, ***J*** angle shown in *Angles of Polygons and Stars, Take 1* was:

$$\boldsymbol{n}, \boldsymbol{J} \text{ angle} = \frac{(\boldsymbol{n}-2*\boldsymbol{J})}{\boldsymbol{n}} *180° \text{ as long as } \boldsymbol{J} < \boldsymbol{n}/2.$$

Since regular polygons have vertices that are equally spaced from one another, it does not matter which vertex the angle is measured from. The ***n*** vertices will all have the same angle so we can talk about **the** angle of an ***n***, ***J*** star.

As is clear from a quick perusal of the table of angles from ***n*** = 3 to 30 (reproduced on the next page), many of the angles are whole numbers. The polygonal angle (the ***J*** = 1 column) increases as ***n*** increases, but we know that this angle must be less than 180°. Additionally, for a given ***n***, the star angle (row ***n***) decreases as ***J*** increases. These observations suggest two questions:

1. Can two distinct images have the same angle?

Some of the values are close to one another. For example, the pentagon's angle (***n*** = 5 and ***J*** = 1) is 108° and the ***n*** = 24, ***J*** = 5 star has angle 105°. Is it possible that if the table were extended, there would be a star that has exactly the same angle as the pentagon, or could there be another star that has the same angle as the 24, 5 star?

We will find our answer to the first question by considering the second question.

2. Are there values of *n* and *J* that create polygons or stars for every whole number angle?

Are all whole numbers less than 180 possible? Let ***A*** be a whole number angle. If there are values of ***n*** and ***J*** that produce this angle, this must be true:

$$\boldsymbol{A} = \frac{(\boldsymbol{n}-2*\boldsymbol{J})}{\boldsymbol{n}} *180.$$

$$\text{Regrouping:} \quad \frac{\boldsymbol{A}}{180} = \frac{(\boldsymbol{n}-2*\boldsymbol{J})}{\boldsymbol{n}} = 1 - \frac{2\boldsymbol{J}}{n}.$$

$$\text{Regrouping:} \quad \frac{2\boldsymbol{J}}{\boldsymbol{n}} = 1 - \frac{A}{180} = \frac{180-\boldsymbol{A}}{180}.$$

$$\text{Dividing by 2:} \quad \frac{\boldsymbol{J}}{\boldsymbol{n}} = \frac{180-\boldsymbol{A}}{360}.$$

We see that the ratio ***J***/***n*** is a ratio of two parts, both of which are whole numbers. If these two parts are relatively prime then ***J*** is simply the numerator, and ***n*** is the denominator. More generally,

$$\boldsymbol{J} = \frac{180-\boldsymbol{A}}{\text{GCD}(360,180-\boldsymbol{A})}, \text{ and } \quad \boldsymbol{n} = \frac{360}{\text{GCD}(360,180-\boldsymbol{A})}.$$

(In the above equations, GCD(x,y) is the greatest common divisor between x and y.) This means that there is an ***n*** and ***J*** that produces an angle of ***A***.

Take, for example, ***A*** = 105. Then, 75 = 180 – ***A*** and GCD(360, 105) = 15 so ***n*** = 360/15 = 24 and ***J*** = 75/15 = 5, just as we saw in the table. This also implies that two distinct ***n***, ***J*** stars cannot have the same angle.

Consider now the ends of the whole number angle range. What are ***n***, ***J*** pairs for a 1° and a 179° angle?

1° 179 = 180 – 1 and GCD(360, 179) = 1, so ***J*** = 179 and ***n*** = 360. This is a really sharp, *sharpest star*.

179° 1 = 180 – 179 and GCD(360, 1) = 1, so ***J*** = 1 and ***n*** = 360. This 360-gon looks a LOT like a circle!

Angle in degrees of Regular Polygons and Stars, 3 to 30

	Polygon	Star jump value ***J*** (***J*** and ***n*** have no common factors greater than 1, and **J** < **n**/2)												
n	(***J*** = 1)	2	3	4	5	6	7	8	9	10	11	12	13	14
3	60													
4	90													
5	108	36												
6	120													
7	128.57	77.14	25.71											
8	135		45											
9	140	100		20										
10	144		72											
11	147.27	114.5	81.82	49.09	16.36									
12	150				30									
13	152.31	124.6	96.92	69.23	41.54	13.85								
14	154.29		102.9		51.43									
15	156	132		84			12							
16	157.5		112.5		67.5		22.5							
17	158.82	137.6	116.5	95.29	74.12	52.94	31.76	10.59						
18	160				80		40							
19	161.05	142.1	123.2	104.2	85.26	66.32	47.37	28.42	9.474					
20	162		126				54		18					
21	162.86	145.7		111.4	94.29			42.86		8.571				
22	163.64		130.9		98.18		65.45		32.73					
23	164.35	148.7	133	117.4	101.7	86.09	70.43	54.78	39.13	23.48	7.826			
24	165				105		75				15			
25	165.6	151.2	136.8	122.4		93.6	79.2	64.8	50.4		21.6	7.2		
26	166.15		138.5		110.8		83.08		55.38		27.69			
27	166.67	153.3		126.7	113.3		86.67	73.33		46.67	33.33		6.667	
28	167.14		141.4		115.7				64.29		38.57		12.86	
29	167.59	155.2	142.8	130.3	117.9	105.5	93.1	80.69	68.28	55.86	43.45	31.03	18.62	6.207
30	168						96				48		24	