Playing with Polygons: Interactive Explorations of Mathematical and Artistic Patterns

A book proposal by Stephen Erfle, Dickinson College, July 13, 2021

This book encourages independent mathematical exploration using regular polygons as the platform for analysis. These explorations are based on a series of Excel workbooks created to examine various aspects of regular polygons, but they are supplemented by web materials written by Liam Myles. Each file is created to focus attention on the topic at hand, rather than the nuts and bolts of how the image was created in Excel (or in the web-app). This allows quicker and more fluid exploration of ideas than possible if creating images by hand. Therefore, pattern recognition and mathematical understanding deepens.

Target audience. There are multiple audiences for this book. One could consider this a book of recreational mathematics, not mathematical research or a mathematics textbook. It is certainly useful for those interested in mathematics education. But it would also be helpful in bridging the gap between mathematics and art. Although mathematics and art teachers in K-12 could certainly incorporate these materials into their classroom, they were initially developed for independent exploration outside the classroom. The idea was to create materials that were easy enough to use that students would want to explore each model and create their own patterns by manipulating the parameters of each model.

Why Excel and how much familiarity with Excel do readers need to have? The files require Excel to operate, but this does not mean that readers need to know how to use Excel to use the files. Most require nothing more than scrolling values up or down with a scrollbar or clicking on or off a click-box to manipulate the image shown. Because they are so easy to use, the files allow readers to focus on the patterns that result. A companion website provides much of the material without Excel. The two delivery mechanisms complement one another as each has features not available in the other.

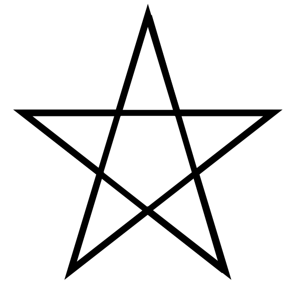
Why regular polygons? Regular polygons are among the earliest geometric figures that students are exposed to and this familiarity often leads to polygons being the template for transformations, reflections, symmetry, counting, etc. Research suggests that students playing with patterns (that can be found in polygons) end up developing stronger math skills.

This book pulls together topics at various levels of mathematical sophistication. Given the highly visual nature of the material, students enjoy exploring images even when they do not fully understand the underlying mathematics that created that image. Such explorations are attuned with Sherman Stein's suggestion that mathematical understanding is best accomplished by following what he called *The Triex: Explore, Extract, Explain* (Stein, 1996). Stein argues that we should encourage our students to explore and gather data, seek to extract some sort of order or pattern from those explorations, and ultimately seek an explanation for what has been found. He points out that even if students are unable to attain that third step, they have been primed to appreciate the explanation for why the pattern exists provided by the instructor or another student (Stein, 1996, p. 6). Such explorations encourage learning and that learning can occur outside the classroom even in the absence of complete understanding of the results.

Layout of the book. The book will have an introductory portion of one to two chapters followed by three major parts. **PART I** is based on string art, sometimes called aestheometry. This part is more highly artistic. **PART II** is based on parallel lines that one can create using the vertices of the polygon and is more strongly mathematical in the sense that it will focus attention on counting rules. (File 6 provides a clear example of what this means.) **PART III** has elements of both **PARTS I** and **II**.

Each chapter will highlight topics and questions that are best examined via a specific file, but the files are scaffolded so that experience with parameters in one file helps in later files. In many instances multiple versions of a file are provided to examine specific issues. I envision each chapter to be presented in a series of short documents (rather than as a large amount of prose). Such documents provide access to materials in the most readily digestible fashion for classroom or independent exploration. Each chapter will include challenge questions that ask the reader to analyze an image or compare across images. File 2, which introduces string art, is an exemplar of the types of documents and challenge questions envisioned here.

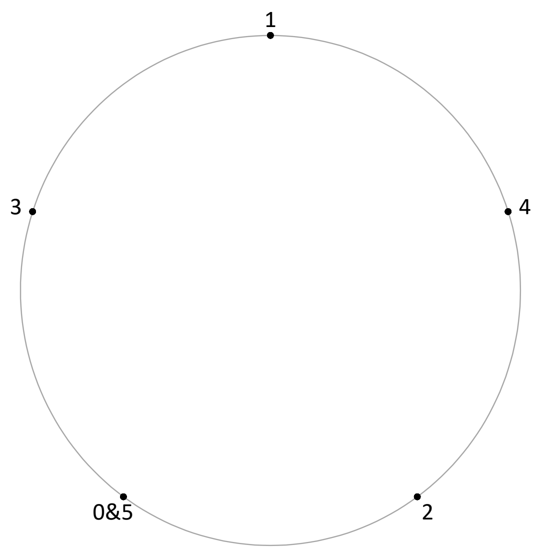
The introductory portion lays out polygon and star basics. We will focus on continuously drawn stars: images created by drawing from vertex to vertex without lifting the pencil until the initial vertex is once again used. Once that occurs, the circuit is complete.

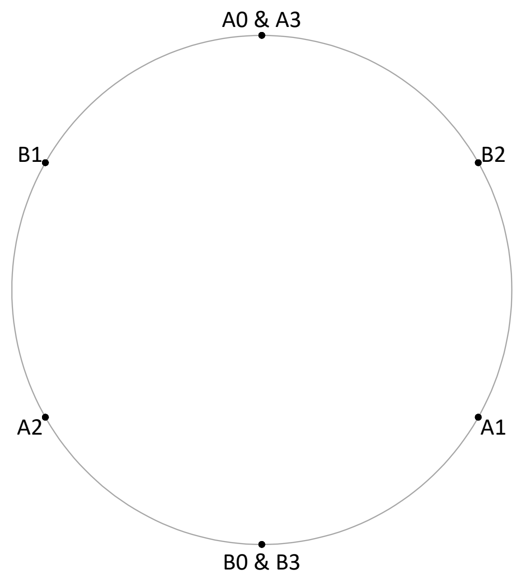


The simplest example of a star is a *pentagram*, or five-point star. A casual survey of more than 100 college students (which included a number of international students as well as students from various parts of the country) were asked without further prompting to draw a star. Then they were asked to think about where they started (Top, Upper Left, Upper Right, Lower Left, Lower Right) and whether they drew the star in a clockwise or counter-clockwise fashion. There are ten possible answers (5 points $\times$ 2 directions).

The results were surprising: more than 80% of students sampled drew a star *exactly* the same way. They started at the lower left then moved clockwise around the circle, jumping over the next vertex (upper left) to draw the first line to the top. The pattern followed is shown on the left where you simply have to follow the numbers from 0 to 5 to draw the star.

The only n-sided polygon with $n > 4$ where one cannot create a continuously drawn star using all vertices is $n = 6$, a hexagon. The problem is shown on the right below. One can draw a 6-point star but only by completing two separate circuits, *A* and *B*. (By contrast, there are two continuously drawn 7-point stars but only one 8-point star.) This and other topics are examined in the introductory part of the book.





Organization of Files in *Playing with Polygons*

1. Polygons Basics (based on EWEP, 2021, "Connecting Geometric Patterns to Numeric Patterns using the *Polygons and Stars* Excel File," *Spreadsheets in Education*, targeted K-2)

PART I. Aestheometry (electronic string art) on Polygons and Stars

2. Introduction to Aestheometry on Polygons (single vertex jump) and Stars (multiple jumps)
3. Center-pointed flowers
4. Two Jump Patterns
5. Four color model with 1 to 3 jump patterns on dodecagon (an extension of E&E, *Bridges 2020*)

PART II. Using polygonal vertices to frame parallel lines

6. Sharpest isosceles triangles on odd polygons (based on E&C, "*Alternative Visions of Perfect Squares*," targeted 3-5)
7. Sharpest right triangles on even polygons
8. Sharpest isosceles triangles on even polygons
9. General triangular model

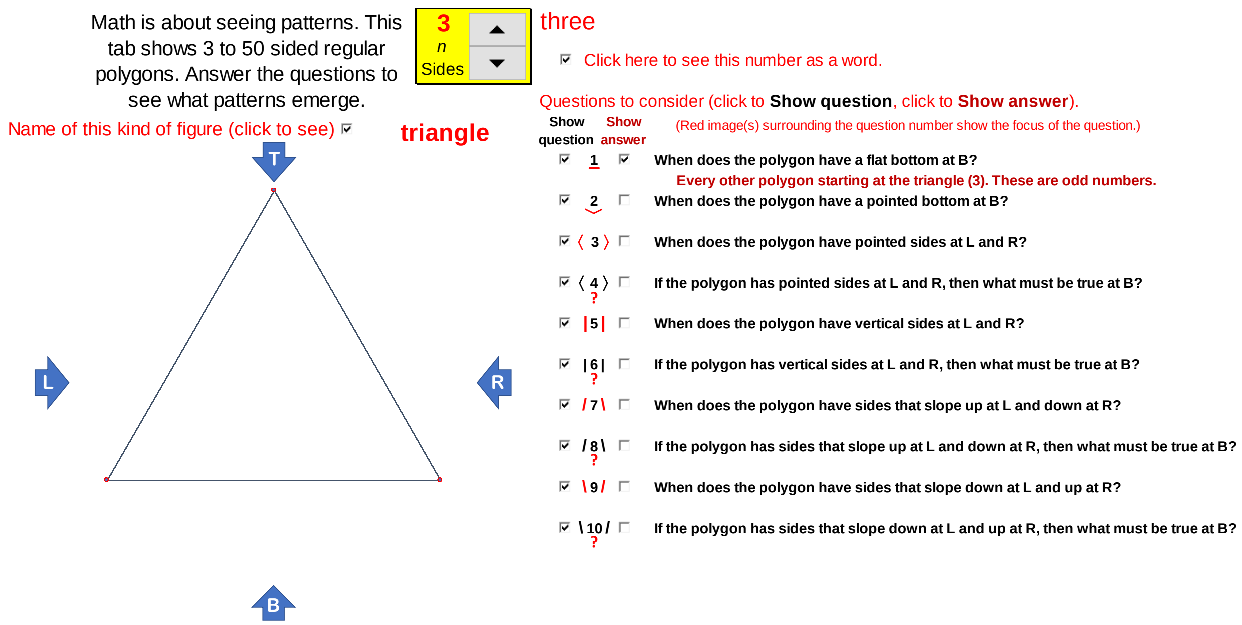
PART III. Images without Subdivisions

10. Spirals (based on "Using Archimedean Spirals to Explore Fractions," *Bridges 2021*)
11. Cardioids
12. Stacked Stars

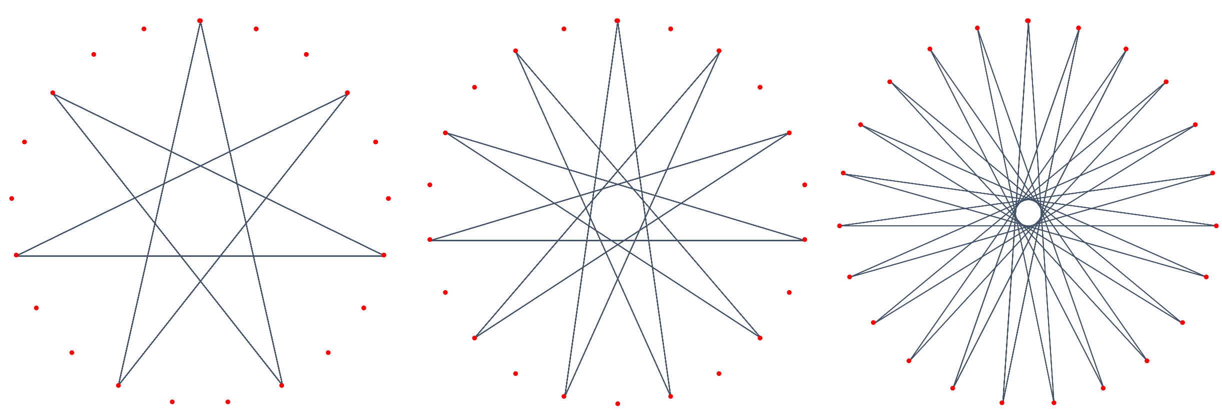
1. Polygon basics: Based on Erfle, Wensel, Erfle and Polinka “Connecting Geometric Patterns to Numeric Patterns using the *Polygons and Stars* Excel File,” *Spreadsheets in Education*, 2021, https://sie.scholasticahq.com/article/21267-teaching-number-patterns-with-polygons-and-stars

Abstract: This paper describes a novel approach for exploring counting and pattern recognition in early elementary classrooms that ties geometric patterns to numeric patterns. Active-learning exercises that start out face-to-face (or using the virtual *Class Circle* sheet of the *Polygons and Stars* Excel file) can be explored quickly and more fully using the *Polygons* and *Stars* sheets. This file can be used by the teacher in the classroom, as well as by students undertaking independent explorations. Such explorations help energize young learners; as they explore, extract, and explain what they have found, they begin to recognize the beauty inherent in mathematical patterns.

The *Polygons* sheet has 10 hideable questions (with answers). The parameter *n* is the number of sides to the polygon.



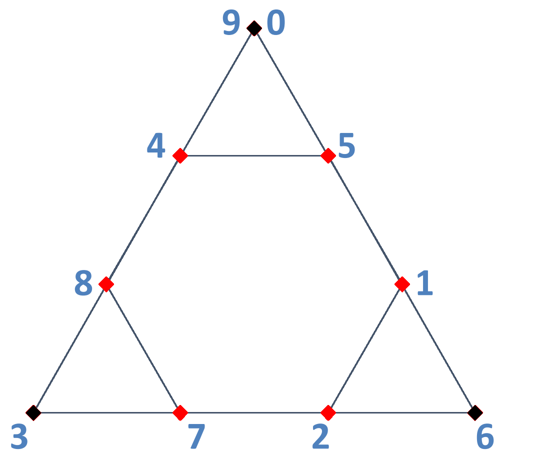
The *Stars* sheet shows what happens when you jump over vertices. Below are 3 examples of stars with *J* being the number of jumps (counted clockwise) between successive lines (*J* = 12 for *n* = 21, 22, and 23)



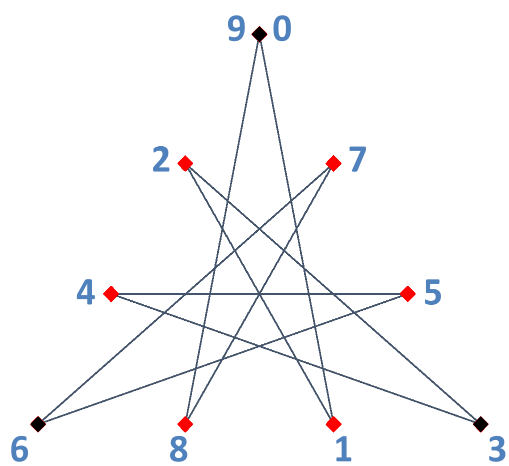
The *Stars* sheet allows users to make continuously drawn stars from 5 to 50 points. All polygonal vertices are used when the greatest common divisor (GCD) between *n* and *J* (image on right) is 1, but when a common divisor exists, the number of points to the star is *n*/GCD(*n*, *J*). GCD(*n*, *J*) is 3, 2, and 1 above so that the stars have 7 = 21/3, 11 = 22/2, and 23 = 23/1 points respectively. K-2 students do not know about division, but they can see that only ⅓ and ½ of the vertices are used in the first two images and they can count points on the resulting star. As noted above, a continuously drawn star can be created for all *n* > 4 except *n* = 6. To create a 6-point star one needs two equilateral triangles rotated 180° from one another.

2. Introduction to Aestheometry on Polygons and Stars: An extension of Polygons and Stars introduces the distinction between *subdivisions between vertices*, *S*, and *subdivisions between points*, *P*. There are 3 subdivisions (*S* = 3) for the two triangular (*n* = 3) frames on the left below, the difference is *P* = 2 for the top image and *P* = 4 below. Follow the numbers from 0 to 9 in both to see how both patterns emerge.

Follow the numbers 1 to 9 in the figures below to see how ***P*** works using ***S* = 3** and ***n* = 3**.

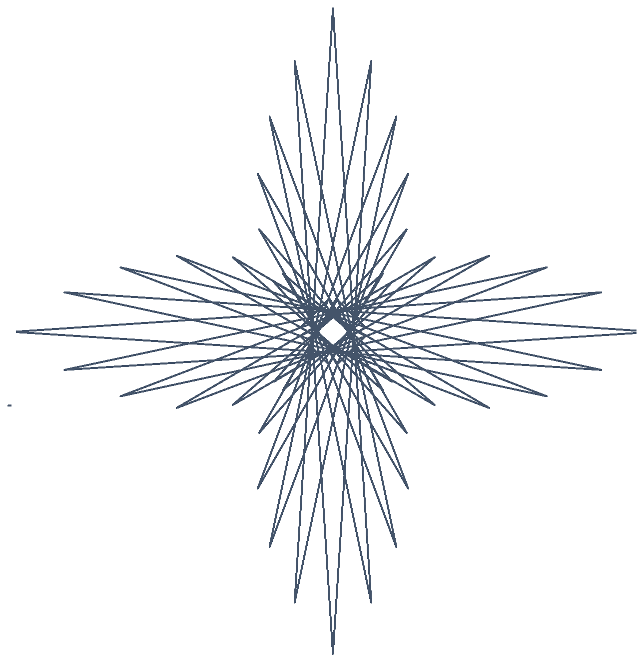


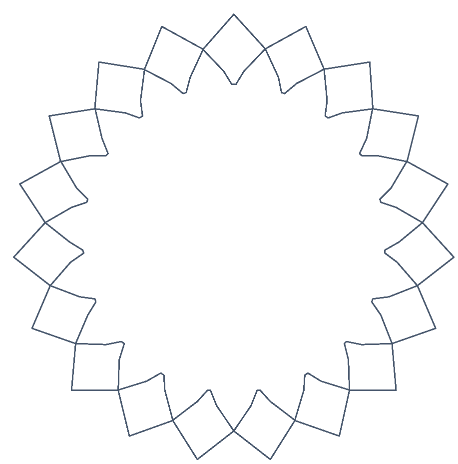
This is ***P* = 2** just like on CurvedTipStars1



This is *P* = 4

Use the up and down arrows to create your own design like the one below (***S* = 13**, ***P* = 12**, ***J* = 23** and ***n* = 48**)





7 *S*, # of *Subdivisions* between vertices	**27** *P*, # of subdivisions between *Points*	**10** ***J*** **vertex jumps between points**	**38** ***n*** possible vertices

Here you control *S* and *P* independently

☐ **Show (*S*,*P*,*J*,*n*) on image**

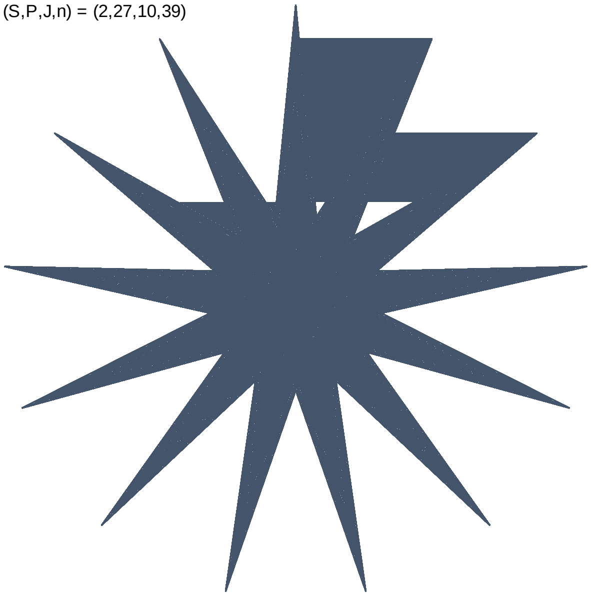
☐ **Show vertices and labels**

NOTE: **If the image looks incomplete, move S or P up or down**

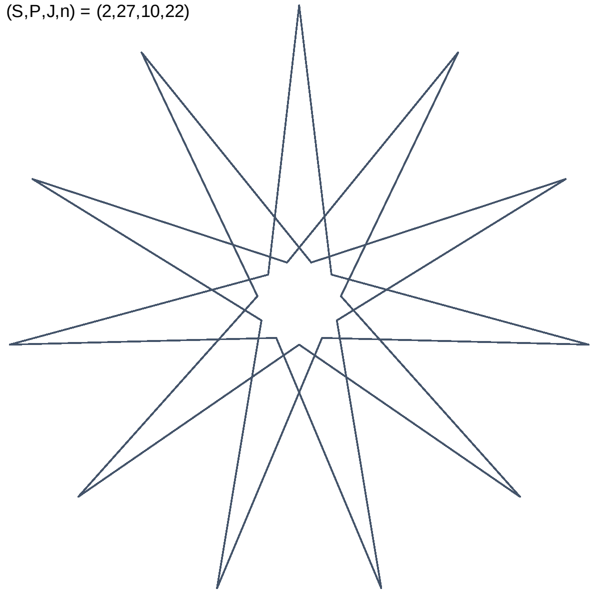
Number of lines in the image, *Lines*	133
Vertex Common Factor, *VCF*	2
Subdivision Common Factor, *SCF*	***1***

These are the dashboard values that produced the lower right image which has 19 petals with 7 lines on each petal (GCD(38,10) = 2, 19 = 38/2, and (7*19 = 133). Change *n* from 3 to 50 and see what happens. The first aestheometry sheet introduces *S* (*P*(*S*) is provided in that sheet); the second aestheometry sheet allows *S* and *P* to be independently controlled. The images on the next page are simply examples showing interesting outcomes. For each, it is instructive to adjust each parameter and see what happens. Notice the star inside a star aspect to the first three images.

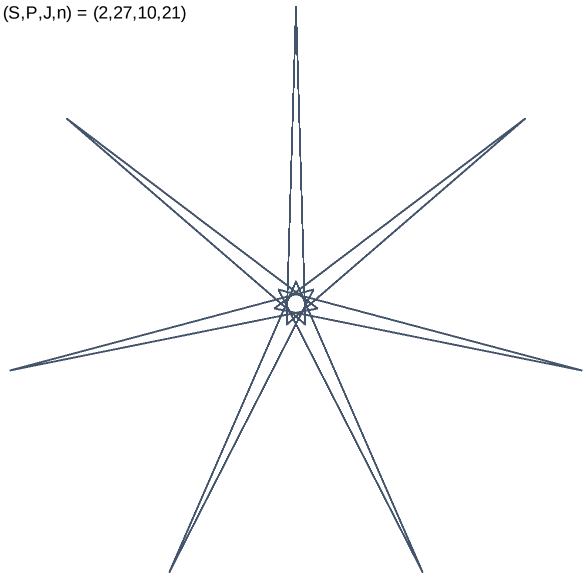
(S,P,J,n) = (2,27,10,39)



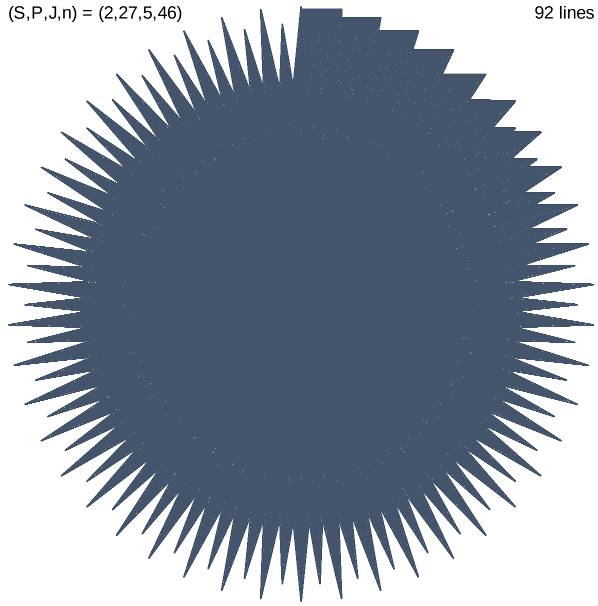
(S,P,J,n) = (2,27,10,22)



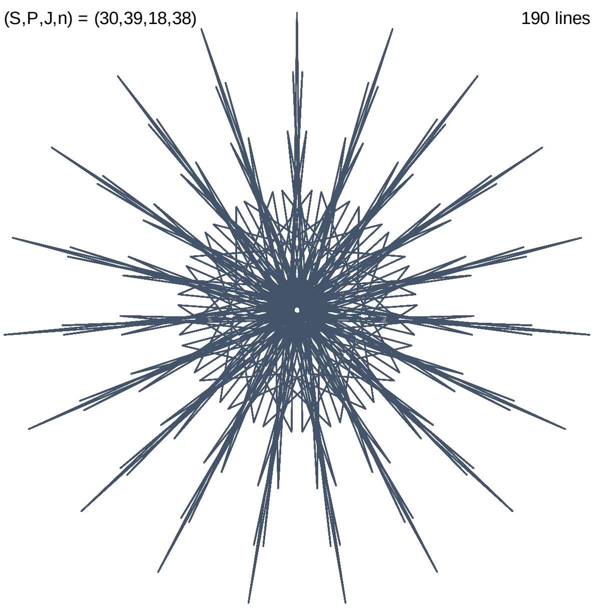
(S,P,J,n) = (2,27,10,21)



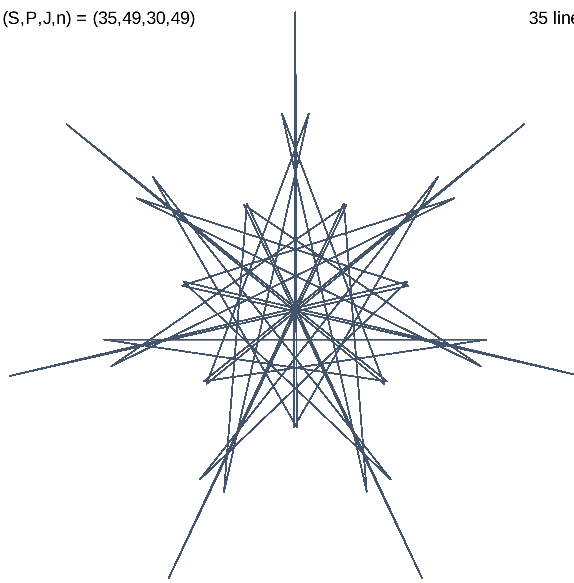
(S,P,J,n) = (2,27,5,46) 92 lines



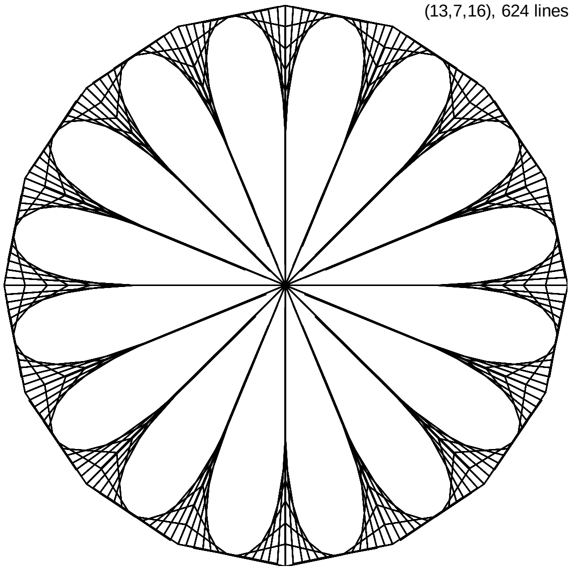
(S,P,J,n) = (30,39,18,38) 190 lines



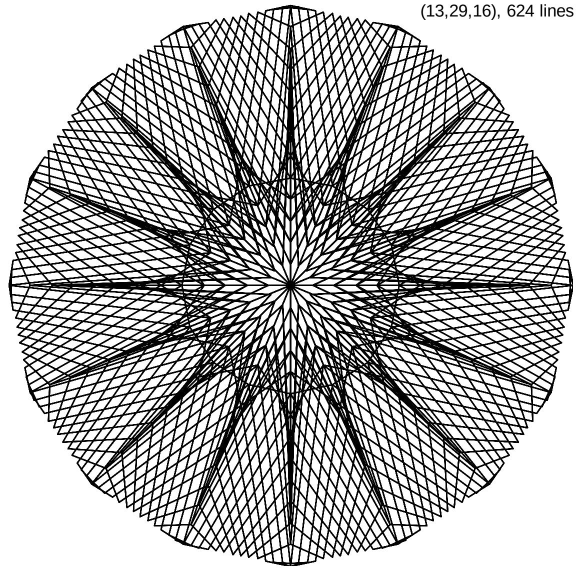
(S,P,J,n) = (35,49,30,49) 35 lines



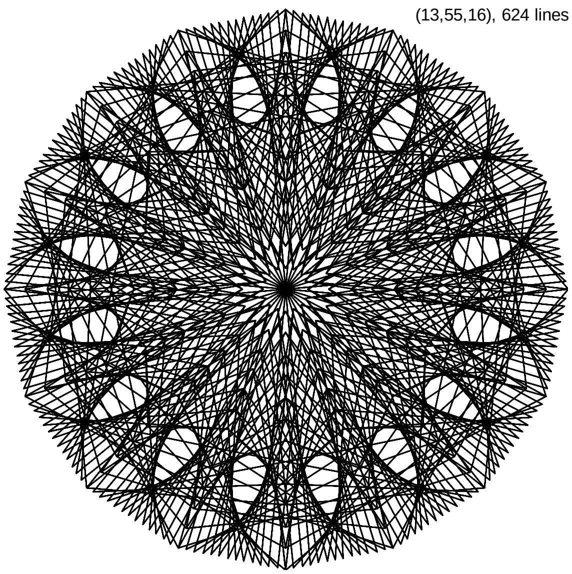
3. Center-pointed flowers. This file treats the center of the circle as an additional vertex between successive polygonal vertices. The resulting (S, P, n) image is a flower when $P < S$. The images below vary P given S = 13 and n = 16. When P = 311, there is a 312 point starburst but when P = 312 (half of 624), a single line appears.



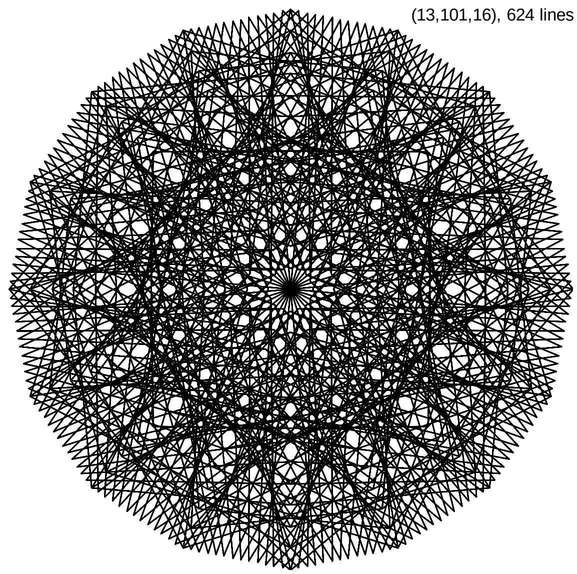
(13,7,16), 624 lines



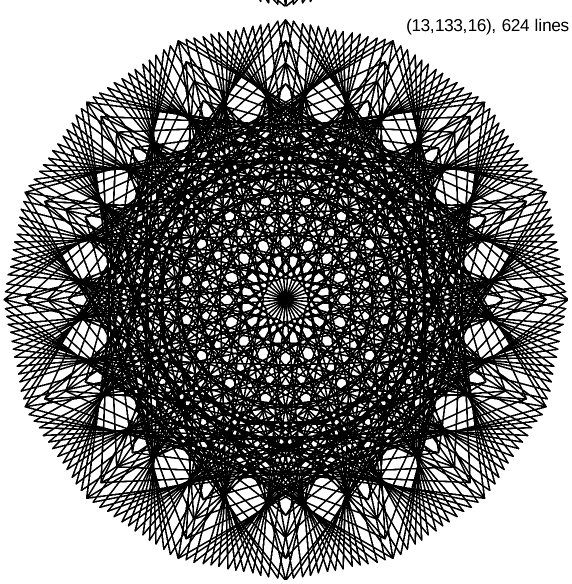
(13,29,16), 624 lines



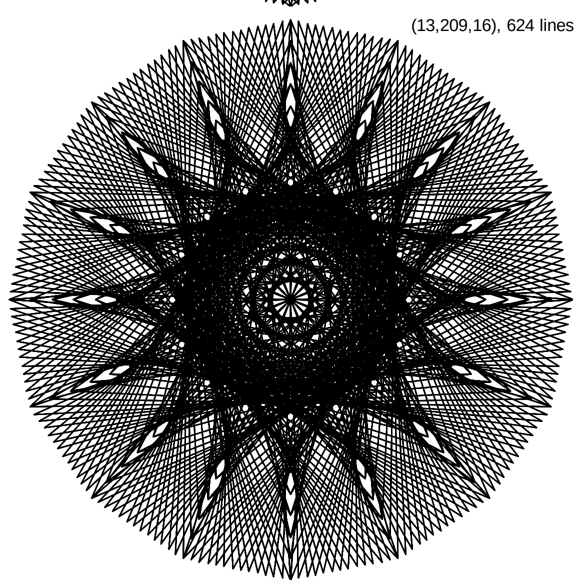
(13,55,16), 624 lines



(13,101,16), 624 lines

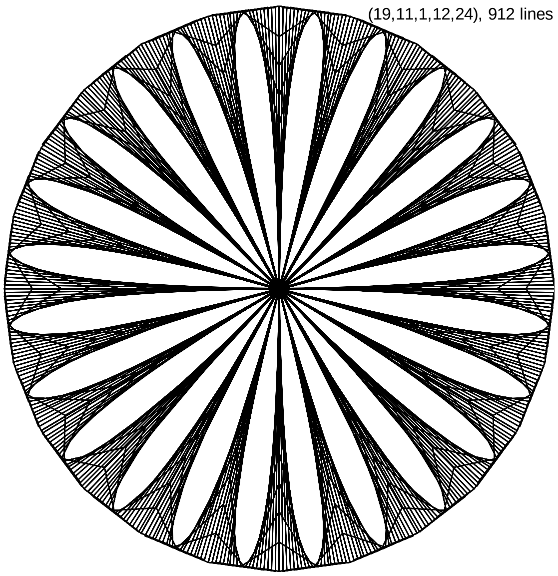


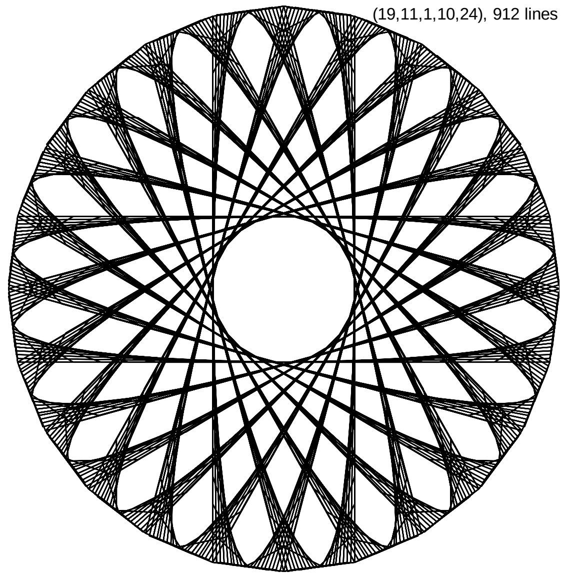
(13,133,16), 624 lines

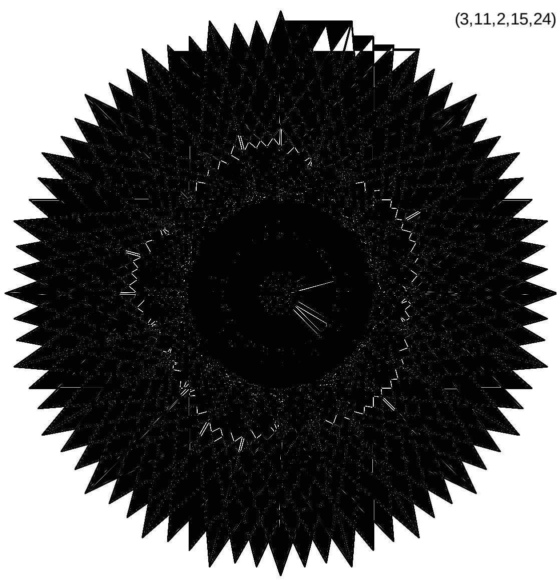


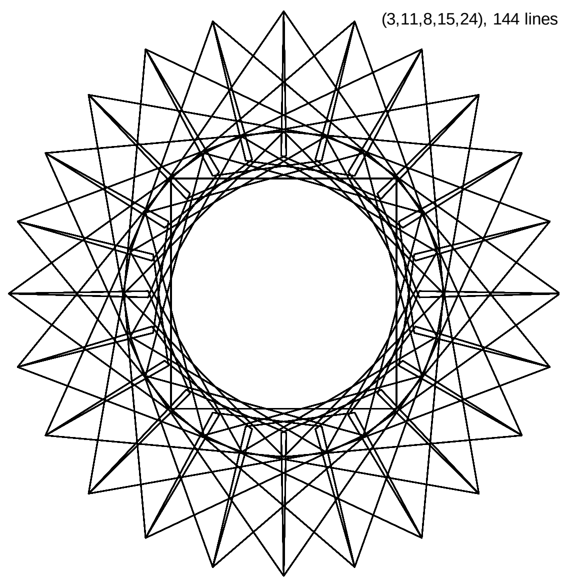
(13,209,16), 624 lines

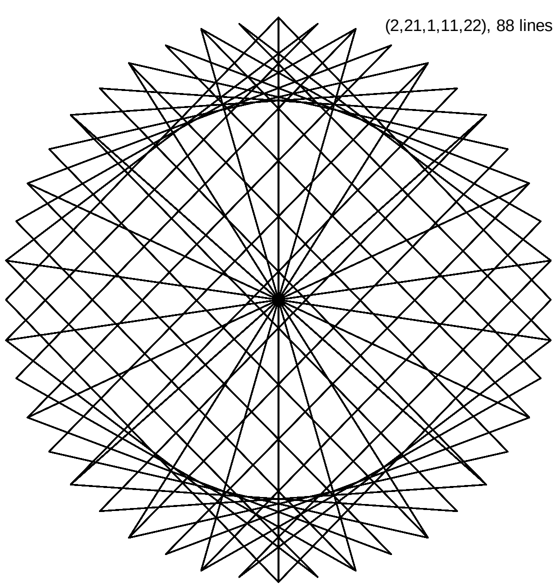
4. Two Jump Patterns. This allows a second level jump, the first image shows how to create flower images (S,P,J_1,J_2,n) when n is even and one of the jumps is $n/2$ and P < S without using the Center-point model. More complex images occur when $P > S$ and if one of the jumps is not $n/2$.

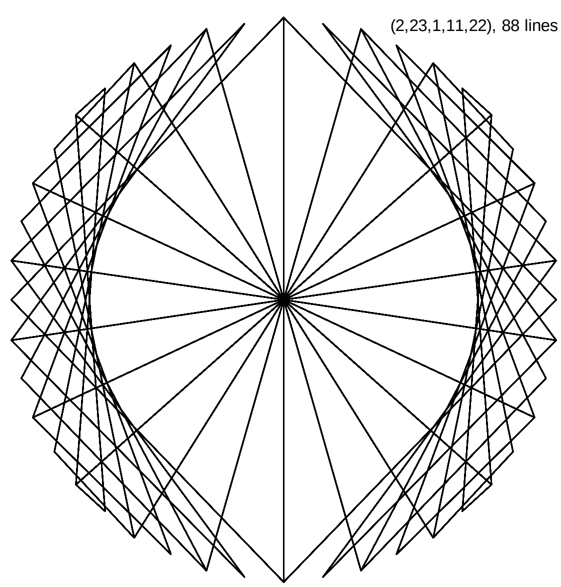




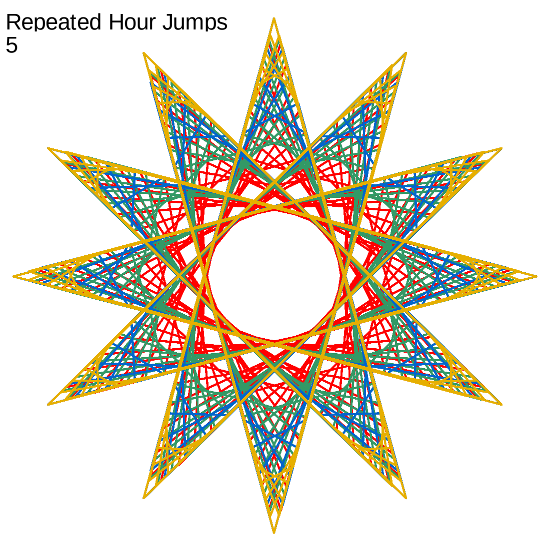


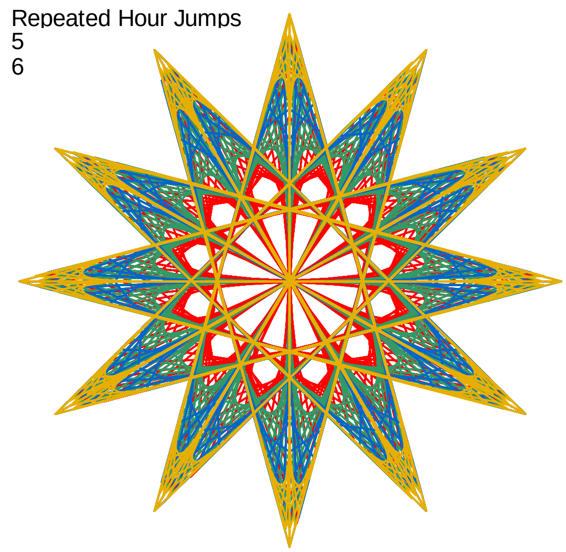


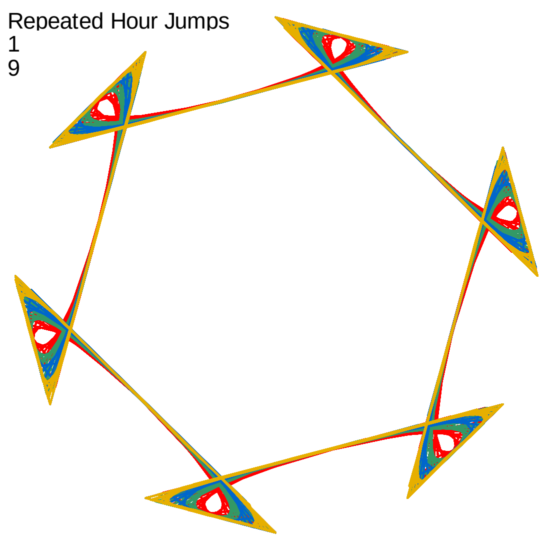


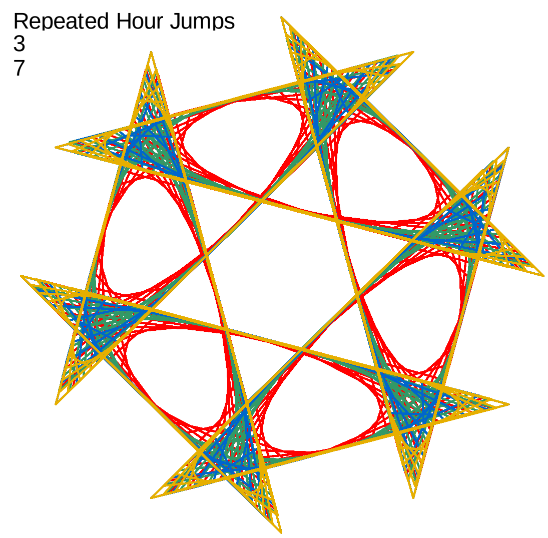


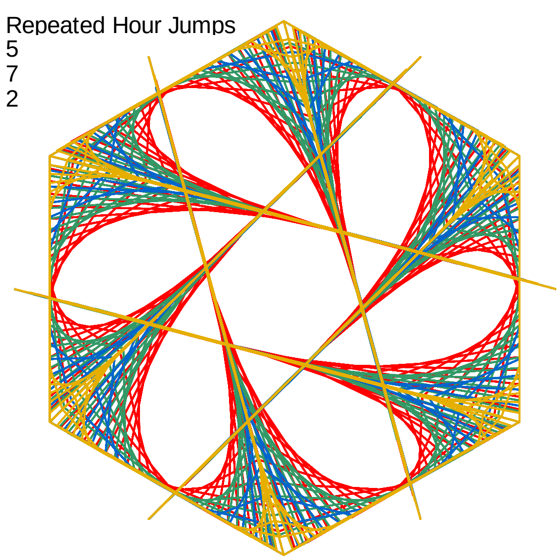
5. Four color model with 1 to 3 jump patterns on dodecagon (an extension of E&E *Bridges 2020* article). The images below provide a glimpse into the wide array of images that can be examined using this model. The model is restricted to a dodecagon because it makes it is easy to talk about the vertices as hours on the clock. The first is 1 jump, the next 3 images are 2 jumps, and the bottom 2 images are 3 jump models. The spikes on bottom left hexagon is the result of the fist to jumps totaling *n* (5+7 = 12) and then jump two vertices over and start again. Had the last value been 3 or 4, a spiked square or triangle results.

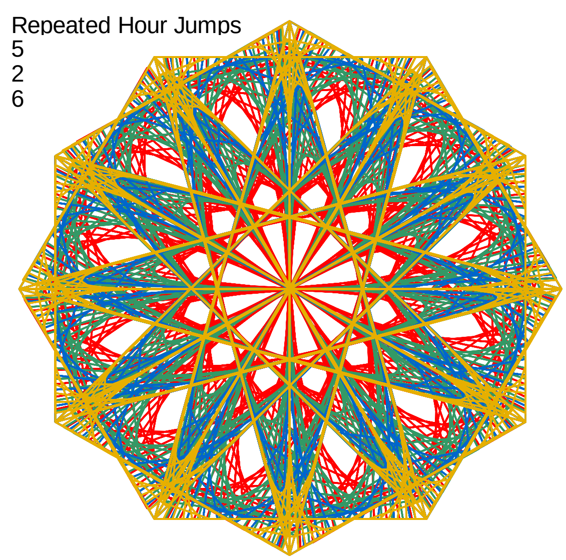










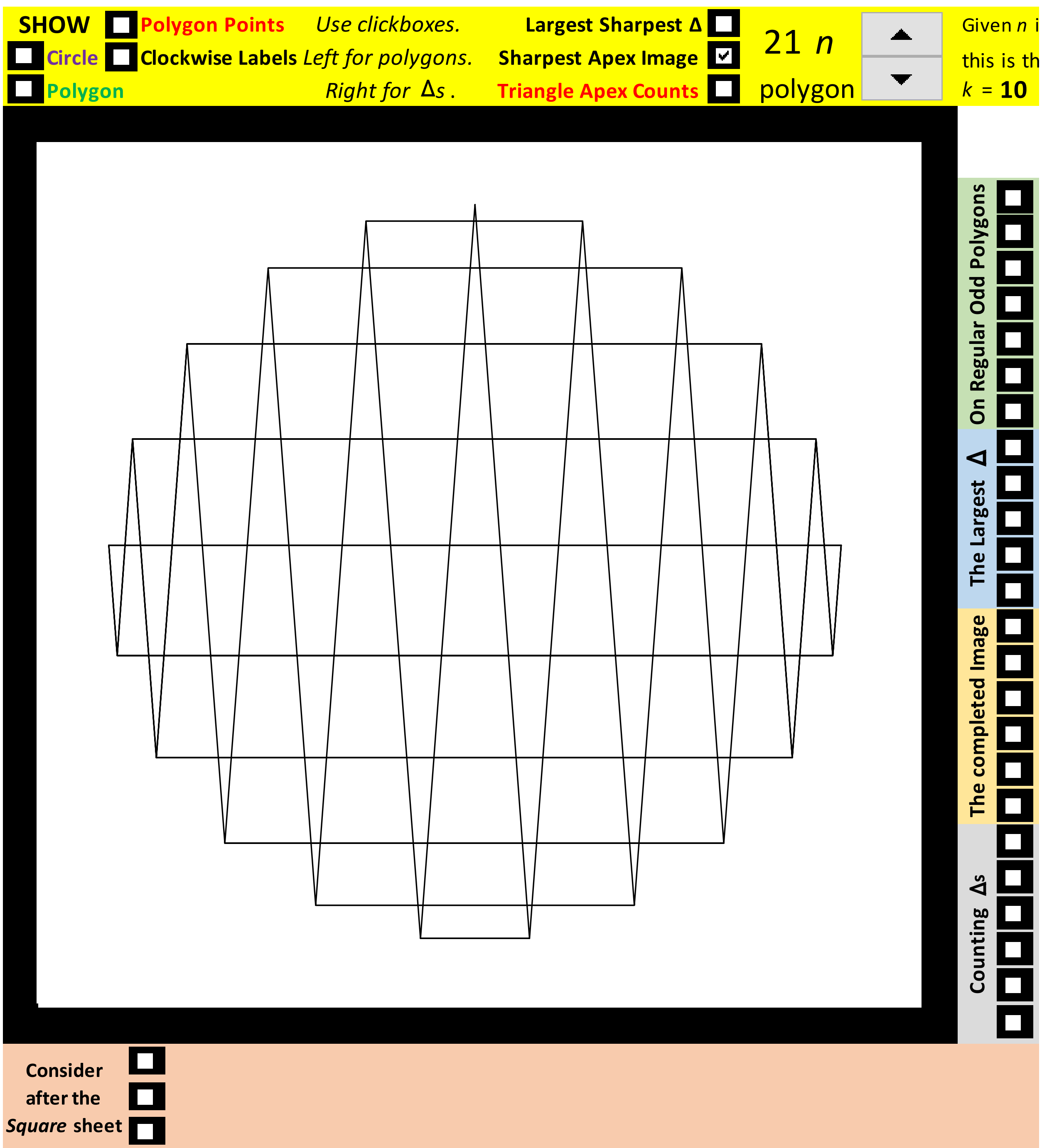


6. Sharpest Isosceles Triangles on Odd Polygons (based on E&C "Alternative Visions of Perfect Squares")

> Abstract from paper: Once students know multiplication, they tend to have a firmer grasp on perfect squares than other multiples. This paper keys off that knowledge and shows multiple geometric and numeric patterns in what comprises a perfect square. The accompanying Excel file is a self-contained teaching tool for in-person and remote classrooms.

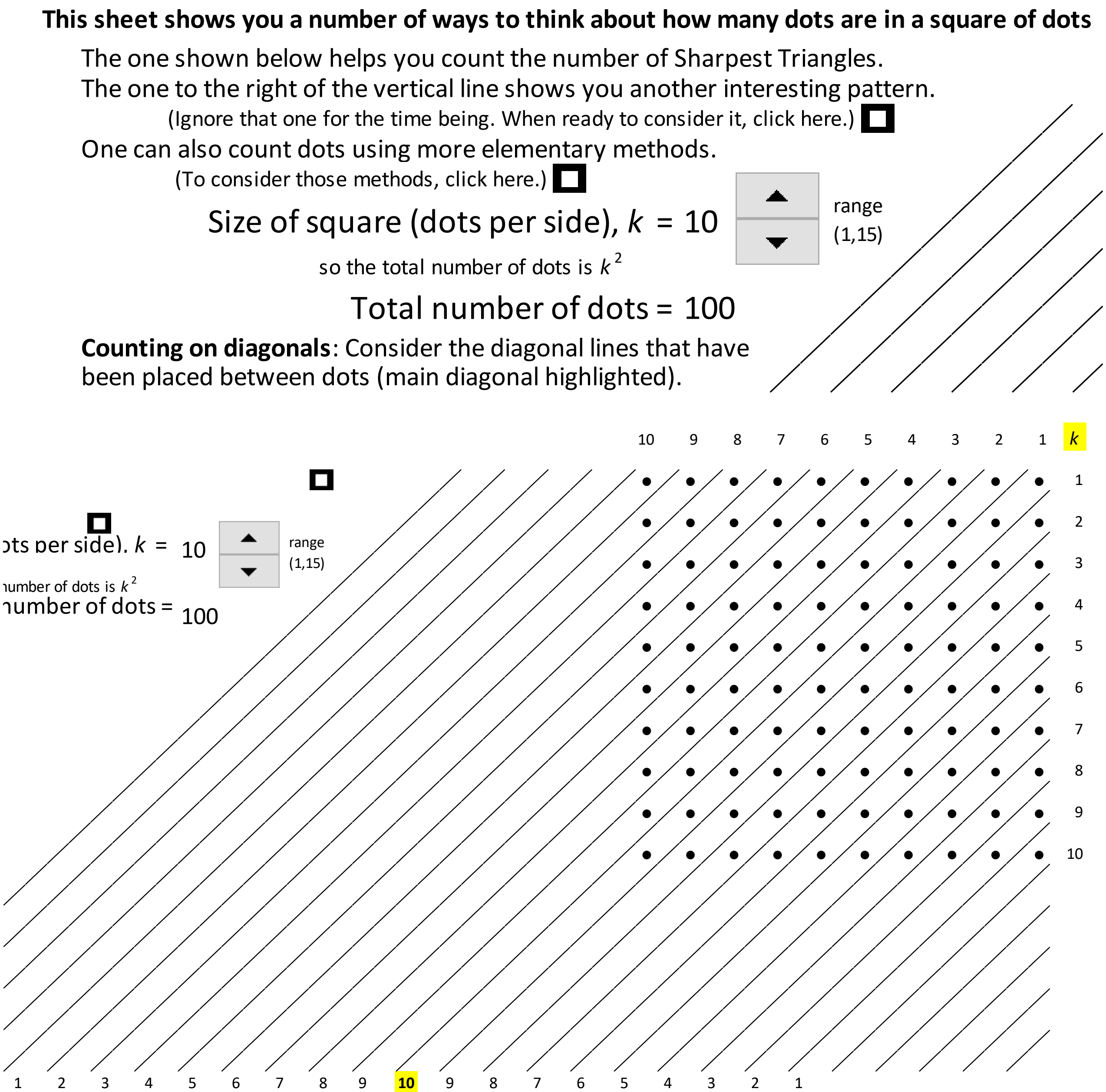
The figure below shows an image with a large number of sharply pointed triangles. An interesting question is: How many triangles are there in the image? The almost magical answer comes by looking elsewhere. The image on the next page shows a perfect square of dots, looked at in a different way. There you see that 1+2+…+9+10+9+…+2+1 = 10^2 = 100. Now look at the left hand side of the image below and count triangle peaks looking in a zig-zag fashion. Note that the counts are identical to the "up the hill and back down" pattern with the top vertex having 10 triangles, so there are 100 triangles. The same question can be asked for any odd polygon (and the file allows you to see images from 3 to 31).

Creating Sharpest Isosceles Triangles Embedded in Regular Odd Polygons



(From Erfle and Chakerian paper)

Figure 3. Two pieces of the *Square* sheet given k = 10 showing topics and diagonal elements



We can use this same set-up to examine a couple of other important formulas. The first is a second way to see k^2 in the resulting image. It is based on the L-shaped additions when k increases by 1. Each addition adds $2k$-1 dots. This is the k^{th} odd number. Put another way, the sum of the first k odd numbers is k^2 (see Figure 4 below for further discussion). Additionally, we can see that the sum of the first k numbers is $k\cdot(k+1)/2$ (see Table 2 below for further discussion). These formulas are useful in counting different, more complex, images. These are the kinds of images examined in an open-ended fashion using the general triangular model in File 9.

As noted in the abstract, the file is set up to be a self-contained teaching tool. The two tables show the click-able discussion points and questions that are contained on the two main sheets.

Table 1. Clickable *Sharpest Triangles* discussion points and questions in five topic areas

On Regular Odd Polygons	In this figure, each regular odd polygon has a fixed vertex at the top of the circle (click A4 on and I2-I4 off, and use the scroll arrows). Label this top point both 0 and n, and other points just like a clock that has n hours rather than 12 (click C3 on). This means that the bottom of the polygon will be flat and there is vertical symmetry. This symmetry means that there are k paired vertices at the same height in the (x, y) plane. In this instance, all vertices from 1 to k have a horizontal counterpart in vertices n-1 to n-k = k+1 (click C2 and C3 on and A4 off). Put another way, we could draw k horizontal line segments between pairs of vertices. The same thing is true in other directions except that then the k parallel lines are no longer horizontal.
The Largest ∆	Three non-parallel lines can be used to create triangular images. We are interested in looking at vertices which create the sharpest apex angle using these vertices. We are also going to restrict ourselves to isosceles triangles (click W2 on to see apex and base angles). Consider the largest such sharpest angle isosceles triangle having a horizontal base and slanted legs. It will have vertices 0, k, and k+1 -- this is the triangle shown by clicking I2 on (this triangle shares a common base with the polygon).
The completed Image	To create the final image, we connect all other vertex pairs which have lines parallel to each of these three lines. To do this, we must draw the other k-1 parallel lines in these three directions. Once done, the completed image emerges (click I3 on). Scroll n from 3 to 31 and watch how the images develop (click Circle, Polygon, and Polygon Points off for sharpest image). How would you describe what happens each time n increases? The apex angle gets a bit sharper and a new "fold" or "wave" happens with the largest horizontal line just above or below the middle. Notice that the new fold is downward pointing when k is odd and upward pointing when k is even.
Counting ∆s	We wish to count all triangles of various sizes in this image. Call this number of triangles $T(n)$. Move from 3 to 5 to 7 and see what happens (remember, some triangles are "upside down") then think about larger n. There are various ways to do this, but the easiest is to use apex vertices (since all apex vertices are also the polygon's vertices). Therefore, for each polygon vertex we can attach a count of triangles with apex at that point (click I4 on). Notice that as we move toward the side from the top or bottom, apex counts decline by 2 per vertex ... to see why, focus on bases. There are various ways to sum apex counts around the circle, go to the **Square** sheet to see one elegant method.
After Square Sheet	If n is large, and you decide to add numbers of apex counts starting at the top and going clockwise, it will soon become tedious. Instead, start at one of the two vertices with apex counts of 0 located at the end of the wave and follow the zig-zag pattern from one side to the other. Notice the number pattern. From here, $T(n)$ should be clear. It turns out that there is another interesting way to visualize k^2 using something called gnomons. To read about that, go back to the **Square** sheet and click the box in AA4. Finally, click Q6 for two additional methods.

Figure 4. Partial *Square* sheet given k = 10 discussing gnomons

k		**Now, ignore the diagonal lines. Instead, focus on the ∟ shaped dots that are added to the left and on the bottom as k increases.**
	gnomon	The nomenclature for this "L" shaped addition is gnomon.
1	1	To count the gnomon of dots, note that adding a row adds k dots, and adding a column adds k dots, and one of those dots,
2	3	the bottom left corner, is common to both row and column. Each gnomon is therefore of the form 2k-1, where 2k-1
3	5	is the kth odd number. This means that: The sequence of gnomons is the sequence of odd numbers.
4	7	In other words, the sum of the
5	9	first k odd numbers is k^2.
6	11	
7	13	
8	15	
9	17	
10	19	
	100	= sum of the first 10 odd numbers.

Table 2. Clickable notes at the bottom of the *Square* sheet showing another use for the hill formula: Gauss addition

1 The hill formula provides a 'side door' to an even more famous pattern in numbers formula: Suppose you are asked to sum the numbers from 1 to 100?

2 If we increase k to 100, the hill pattern would have that sum plus the sum from 99 to 1. If we add 100, we have twice the sum from 1 to 100. Therefore, $100^2 + 100$ = twice the sum from 1 to 100.

3 Dividing by two we have: The sum from 1 to 100 = $(100^2 + 100)/2 = 100\cdot(100+1)/2 = 5{,}050$.

4 More generally, $1 + 2 + \ldots + k = k\cdot(k + 1)/2$. This is an example of Gauss addition (click next box to learn more).

<u>The highlighted material may be too difficult for Grades 3 and 4.</u>

5 The classic story goes that Carl Friedrich Gauss recognized a pattern as a young child when asked to sum the numbers from 1 to 100.

6 He noticed that if you take a second copy of those numbers and reverse their order and put them on top of one another, something magical occurs. To see this click the next box.

7

$$\begin{array}{r} 1 + \quad 2 + \quad 3 + \ldots + \ 98 + \ 99 + 100 \\ 100 + \ 99 + \ 98 + \ldots + \quad 3 + \quad 2 + \quad 1 \\ \hline \end{array}$$

Instead of adding horizontally, add vertically: $101 + 101 + 101 + \ldots + 101 + 101 + 101$

8 Each vertical sum is the same and the top row shows how many 101s are present. Therefore, twice the sum of 1 to 100 is $100\cdot101$ so the sum of 1 to 100 = $100\cdot101/2 = 5{,}050$.

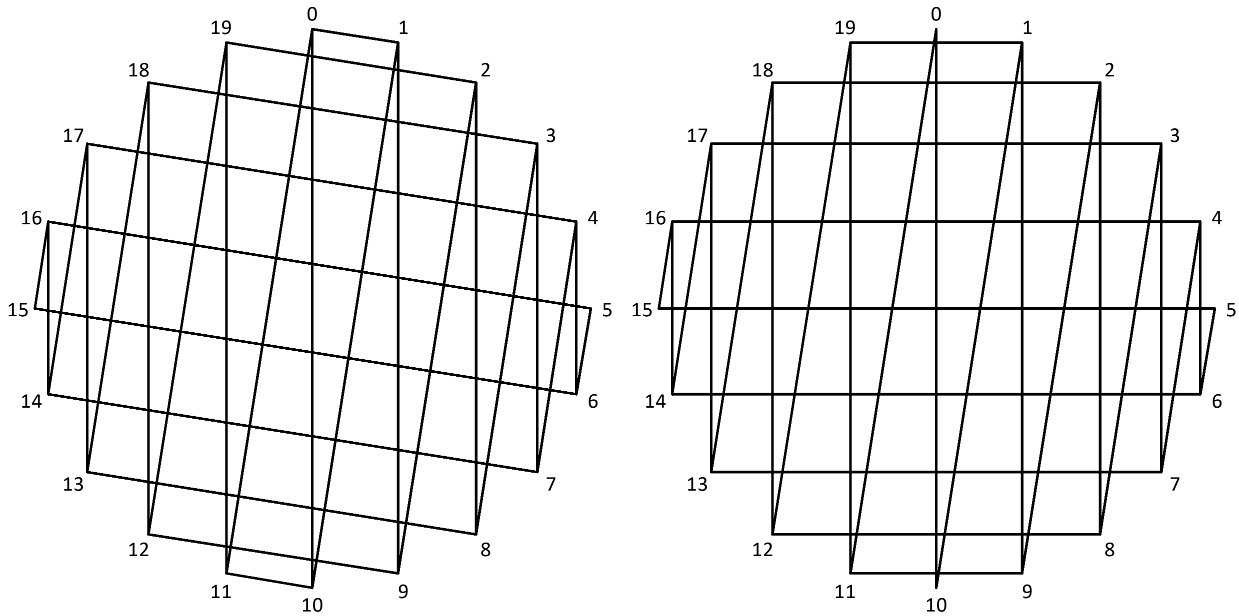
9 It is worth noting that it is standard practice to go in the opposite direction and derive the hill formula (1) from the sum of the first k numbers formula (3) rather than deriving (3) from (1).

7. Sharpest Right Triangles on Even Polygons. (File based on a minor adjustment to File 6.)

When n is even, one can no longer have sharpest angle isosceles triangles because n-1 is not divisible by 2 if n is divisible by 2. Put another way, when *n* is even, a sharpest apex triangle (in which the sharpest angle is formed from two contiguous vertices) cannot be isosceles since *j*+ (*j*+1) is, by necessity, an odd number. But, when *j* = *n*/2 and *k* = *j*+1 the resulting triangles are tall-skinny right triangles like those shown below. (Interestingly, one cannot create right triangles using polygonal vertices when *n* is odd.)

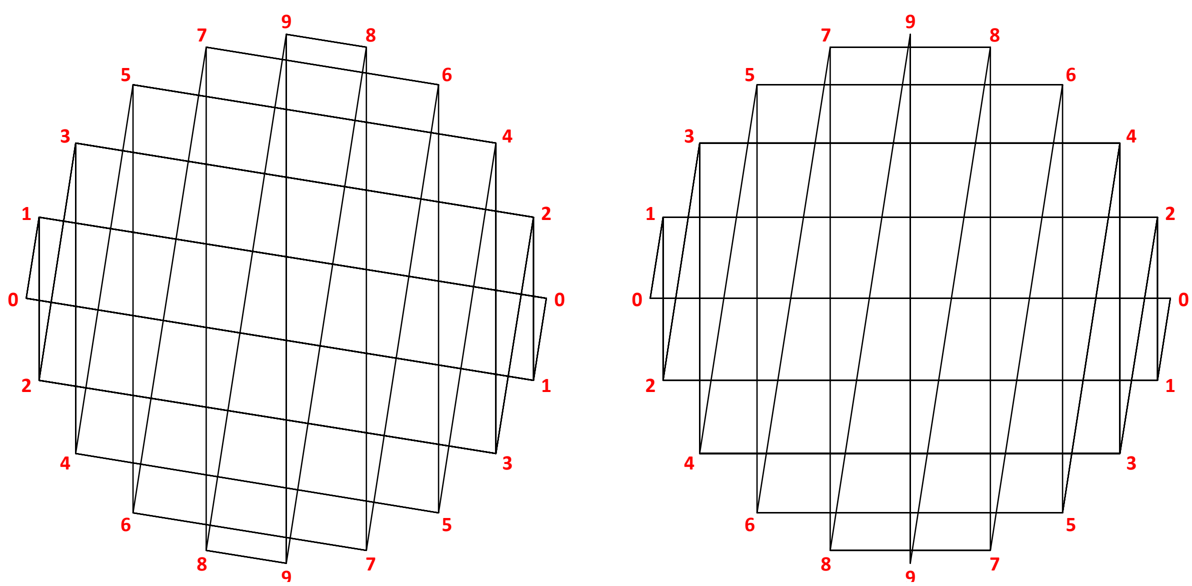
Two versions are possible, panel (a) has slanted sides opposite the apex angle and panel (b) shows the same apex angle given horizontal rather than slanted sides. Alternatively, we can view the difference between panels in terms of the hypotenuse of the largest such triangle: in (a) the largest hypotenuse is the vertical diameter from 0 to *n*/2 while in (b) there are two largest hypotenuses, one from 0 to *n*/2+1, the other from 1 to *n*/2.

Both images have the same apex pattern count of 1+2+…+8+9 + 9+8+…+2+1 = 9·10 = 90 triangles. This counting pattern was discussed in Table 2 and at the bottom of the *Square* sheet in File 6.



(a) Slanted base (b) Horizontal base

Figure 1. Two versions of sharpest apex right triangles



(a) Slanted base (b) Horizontal base

Figure 2. Apex counts for two versions of sharpest apex right triangles

8. Sharpest Isosceles Triangles on Even Polygons (based on Chakerian and Erfle, “Discovering Number Patterns in Triangle Counting.”)

The sharpest isosceles triangles that one can obtain on even polygons has an apex angle that spans two vertices rather than one vertex as seen with File 6. This leads to more complex images because apex vertices are no longer restricted to polygonal vertices as they were with Files 6 and 7. Now a set of interior apexes occur as seen in the images below (taken from the paper). Once again, two versions are possible for each *n* and different counting solutions occur for *n* = 4*k* and *n* = 4*k*+2 polygons.

The first such images came from File 4.2 for *n* = 4*k*+2 polygons (by setting *S* = *Jump 1* = 1 and *P* = *n*-1) but that file can only recreate one of the four possible versions. (In particular, it produces a rotated version of (d) below.)

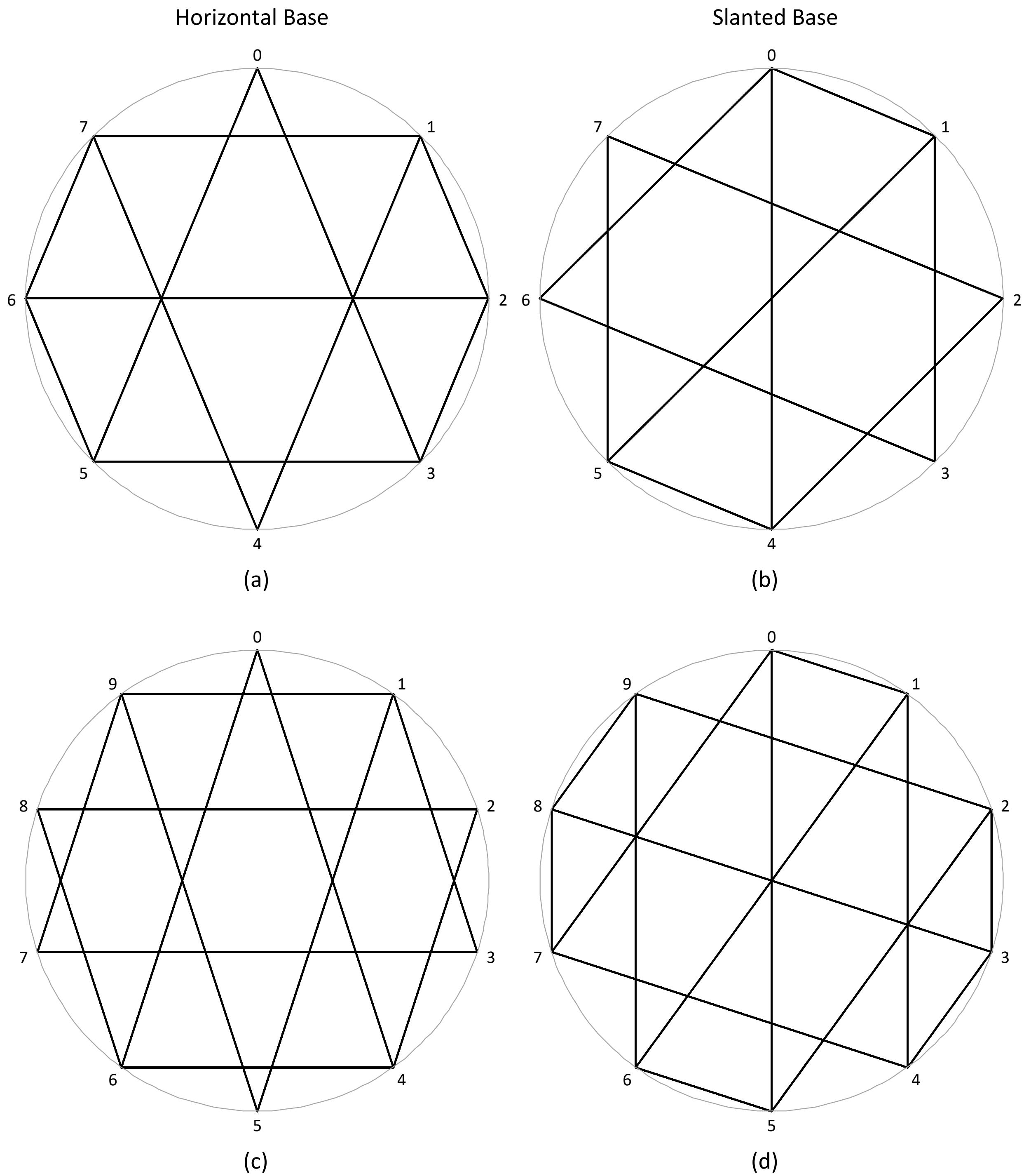
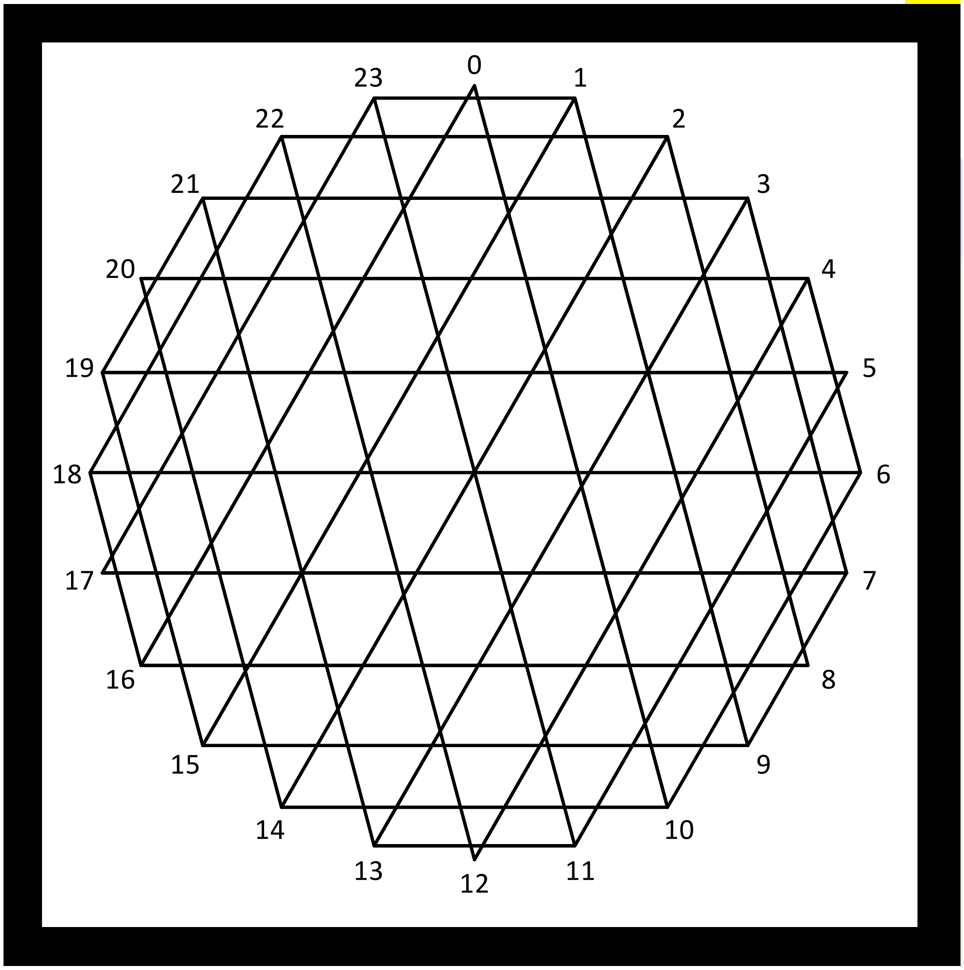


Figure 7. Triangles created when *n* is even: Top row, *n* = 8; bottom row, *n* = 10

9. General triangular model sets up triangular images connecting polygonal vertices in three directions. The following is the dashboard with instructions for using the file.

Create your own triangular patterns on regular polygons, *n* 24



(2 < *n* < 32) **Images are created by defining three non-parallel lines between vertices**

SHOW **Vertex #'s** ☑ ☐ **Points** **For simplicity the first two lines use vertex 0 = (0, 1).**

Circle ☐ **Line 1 is 0 to *j* = 16 *j* = 1, …, *n*-1. Line 2 is 0 to *k* = 10 *k* ≠ *j*, *k* = 1, …, *n*-1.**

Lines ☑ **Line 3: 1st vertex, *v* = 4 *v* = 0, …, *n*-1. 2nd vertex, *w* = 20 *w* ≠ *v*, *w* = 0, …, *n*-1.**

Parameter values ☐ ***Note:* The third line need not include *j* or *k*, although that is fine.**

a, b, c are arcs of circle summing to *n* and represent angles $a/n \cdot 180°$, $b/n \cdot 180°$, and $c/n \cdot 180°$ 6 = *a* = min(|*j*-*k* |,|*j*-*s* |,|*s*-*k* |)

To obtain *a*, *b*, & c from *j*, *k*, *v*, *w*, & *n*: 0 = *s*, The line *vw* is parallel to 0*s* with *s* = MOD(*w*+*v*,*n*) 10 = *b* = *MAX* - *a*

16 = *MAX* = MAX(|*j*-*k* |,|*j*-*s* |,|*s*-*k* |) 8 = *c* = *n* - *b* - *a*

Additional instructions: You can manually enter numbers in the yellow cells, or you can insert equations in those cells. The yellow cells have been labeled so you can refer to them by name. For example entering, = INT(*n*/3) in Q3, = *n*-*j* in V3, 1 in R4, & = *n*-*v* in W4, produces "near equilateral" isosceles Δs. These are exact when *n* is divisible by 3.

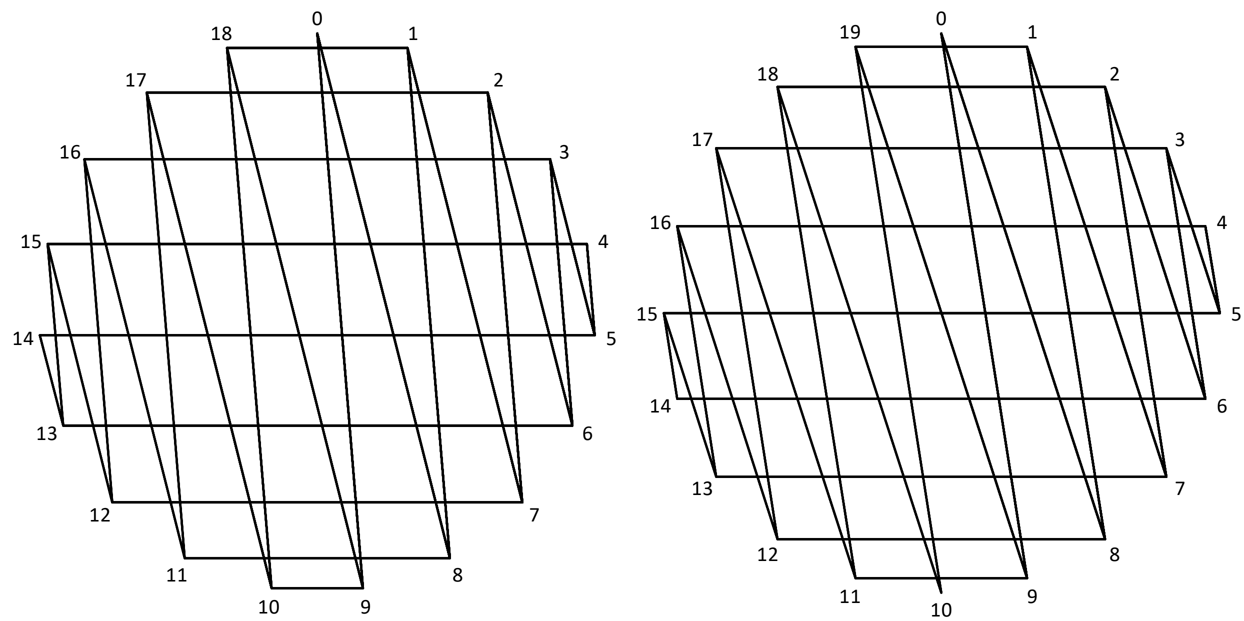
To rotate image: 0 = *r*, rotation factor r = 0, 1, …, *n*-1 using these equations $j = \text{MOD}(j_0+2r,n)$, $k = \text{MOD}(k_0+2r,n)$, $v = v_0$, $w = \text{MOD}(w_0+2r,n)$

To create parallellograms, set *v* = 0 and *w* = *j*.

This file can be used to examine a large array of images. This provides for open-ended exploration based on various rules. For example, the File 6 images of sharpest angle isosceles triangles using odd polygonal vertices are obtained with odd *n* = 2*k*+1 with *j* = *k*+1, *k* = *k* (or (*n*-1)/2 based on *n*), *v* = *j*, and *w* = *k* (the image shown in File 6 above set *n* = 21 or *k* = 10). Beyond that, we might wonder how things change when *n* is even, or when the triangles are not isosceles. Each of these questions and more can be examined by creating and analyzing images using this file.

For example, Files 6 and 7 examined sharpest apex isosceles and right triangles. File 9 can help us examine what happens for the more general situation involving sharpest triangles.

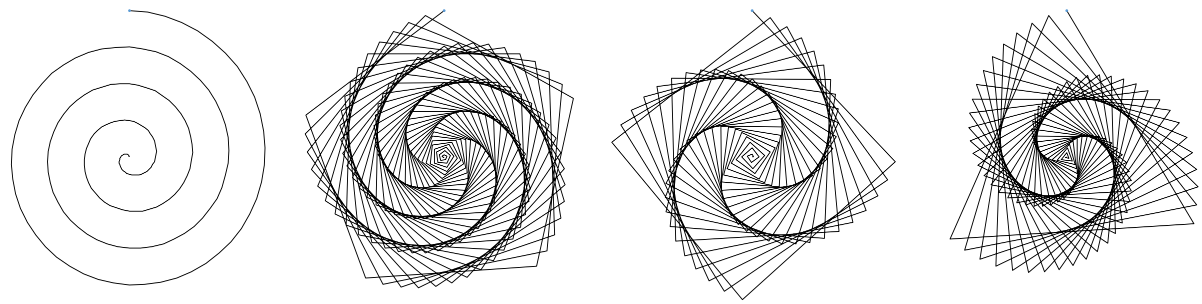
One can readily examine the more general situation with angles $(1, b, c)\cdot(180/n)°$ where $b + c = n - 1$. File 6 examined the case where $b = c$ (which implies n is odd). File 7 examined the case where $|b - c| = 1$ (for even n). The more general case for sharpest apex triangles would have $b \neq c$ and $|b - c| > 1$. In this instance, there will be a smaller number between b and c (let that be b). The zig-zag wave pattern is still visible, but it now plateaus with apex counts of b. The total number of triangles is $T(n, b) = b\cdot(b + 1) + b\cdot(n - 2\cdot(b + 1))$. The triangle count is therefore 80 in Figure 2.a and 88 in Figure 2.b.



(a) $n = 19$, $(1, 8, 10)\cdot(180/19)°$ (b) $n = 20$, $(1, 8, 11)\cdot9°$

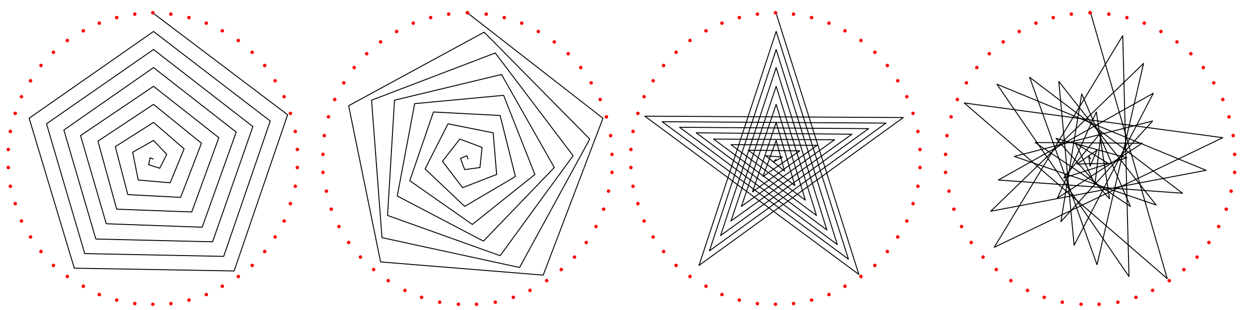
Figure 2. Two versions of sharpest apex obtuse triangles inscribed on n-gons with horizontal base

10. Spirals (based on Erfle, "Using Archimedean Spirals to Explore Fractions," *Bridges 2021*). A modest addition to File 1 produces spirals and swirls. Interesting patterns emerge based on jump pattern, the size of the polygon as well as the reduction in radius, r, per jump. Images shown are from the *Bridges* article. Twisted polygonal images occur when n/J is "close" to an integer and that image rotates in different directions based on whether the fraction is just over or just under an integer value.



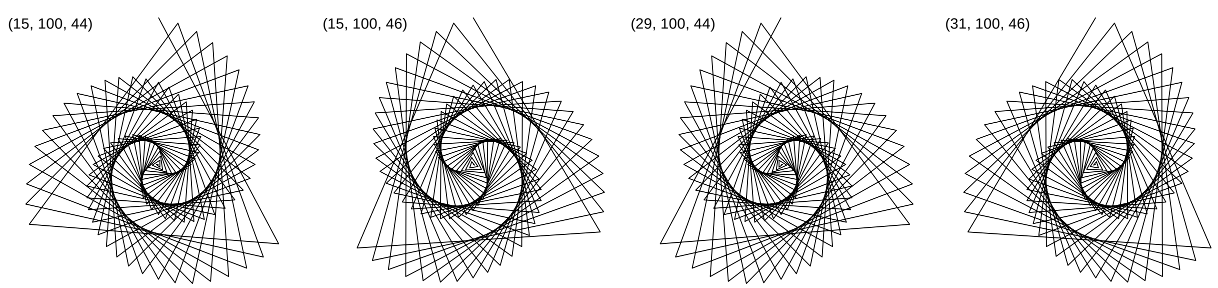
(a) (b) (c) (d)

Figure 1: *Sample images created with the Spirals Excel file.*



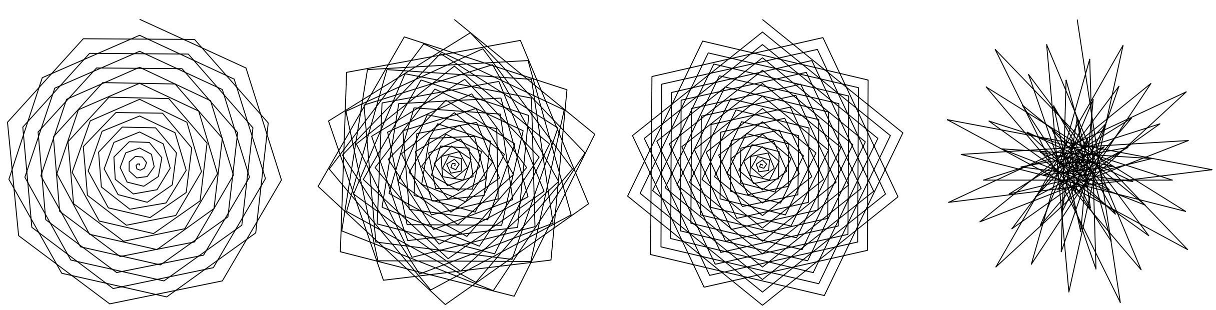
(a) J = 10, n = 50 (b) J = 10, n = 49 (c) J = 20, n = 50 (d) J = 20, n = 49

Figure 4: *Images created with 40 connected line segments, r = 40.*



(a) ↻-spiral, 🌀-swirl (b) ↻-spiral, 🌀-swirl (c) ↺-spiral, 🌀-swirl (d) ↺-spiral, 🌀-swirl

Figure 6: *Comparing Clockwise and Counterclockwise Spirals and Swirls, (J, r, n) values noted.*

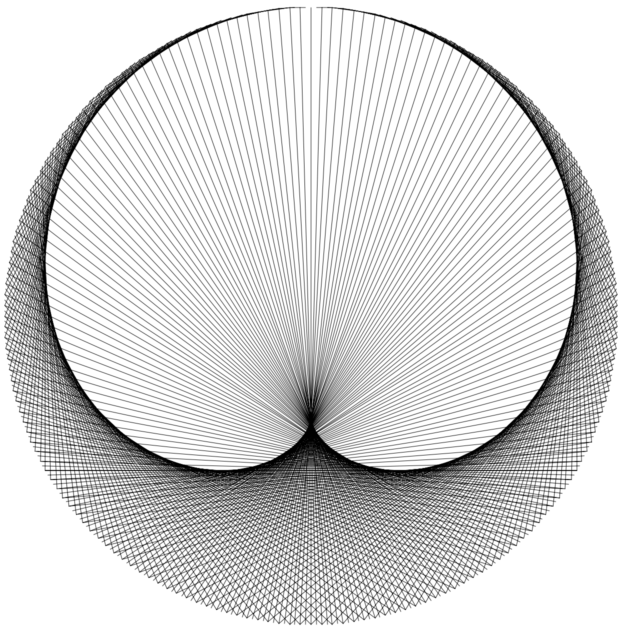


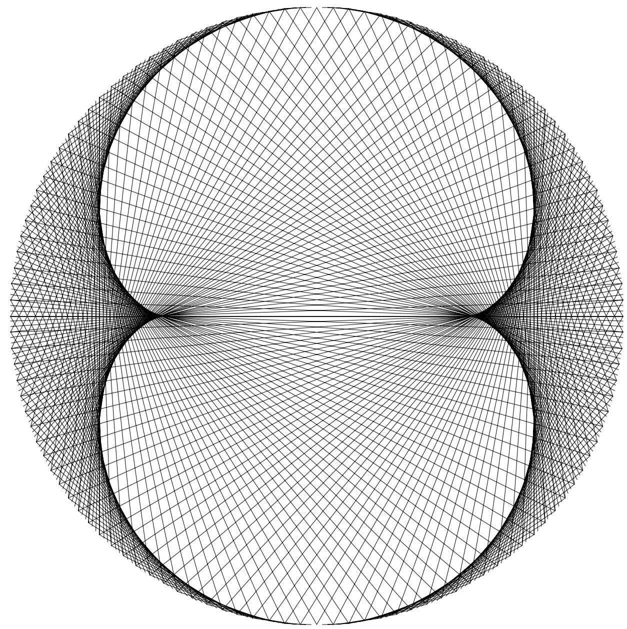
(a) (b) (c) (d)

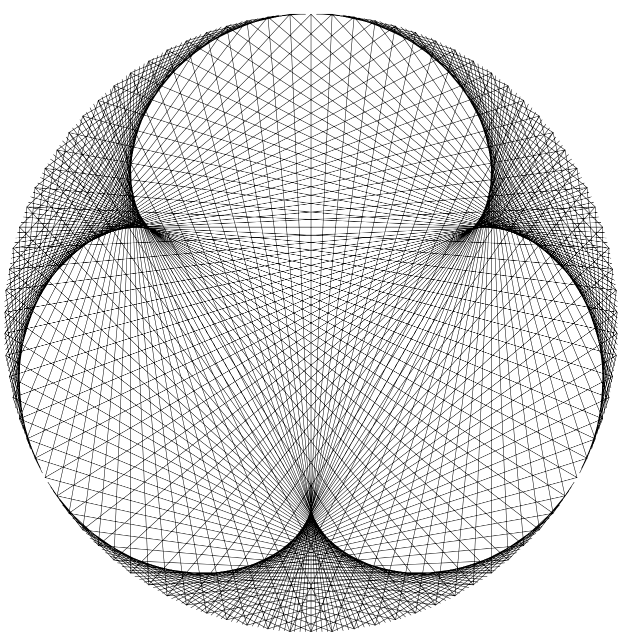
Figure 7: *Examples of flower images.*

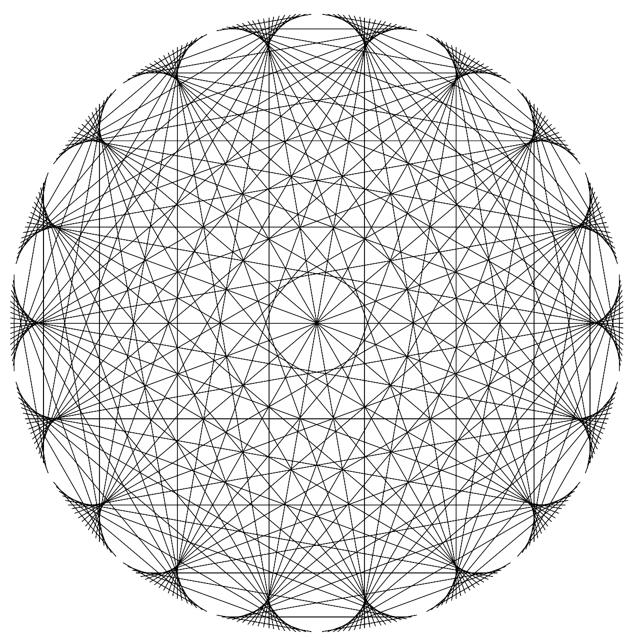
11. Cardioids can be obtained by drawing line segments connecting vertices of the polygon: (1, 2), (2, 4), (3,6) … (j, 2j) from j = 1, …,n-1. Of course, many of the second points will be larger than n in size … these points are simply the MOD n values of 2j (MOD is the remainder upon division by n). (For example, if n = 6 and j = 4 then the rule is to connect point 4 to 8, but 8 is the same as 2 because 2 = MOD(8, 6). Note that this is simply a second time to have drawn that particular line since (2, 4) was already drawn.)

The images below are based on n = 360. The upper left is a classic cardioid. We can generalize this to obtain multiple cardioids around the circle by changing the rule from (j, 2j) to (j, kj). This will produce k-1 gathering points. The remaining three images show k = 3, 4, and 19.









The cardioid images are created much like File 9 which creates general similar triangles using parallel lines in that each of the lines is set independently from others, rather than as a single closed circuit as was the case in **PART I**.

12. Stacked Stars is a closed circuit file, unlike 10 and 11, but like the files in PART I. File 4 examines double jump patterns, and File 5 explores triple jump patterns on dodecagons. File 12 expands the reach to much larger polygons and allows up to 9 distinct jumps per set. Many of the images are reminiscent of doilies. Just like in PART 1, images can degenerate if the set sum of jumps has factors in common with n.

One of the images one can create is a “Mystic Rose” in which each vertex is connected with all other vertices. The version below is a 19-gon mystic rose with 171 = 19*18/2 lines. This rose is drawn as a single circuit because the sum of 1+2+…+9 = 45 and 7 = MOD(45, 19) has no common divisor with 19. Beneath that is the same jump pattern with n = 18 and n = 20. The first completes its circuit in 18 segments due to a common factor of 9 between 18 and 9 = MOD(45, 18); the second takes 36 segments to complete its circuit due to a common factor of 5 between 20 and 5 = MOD(45, 20).