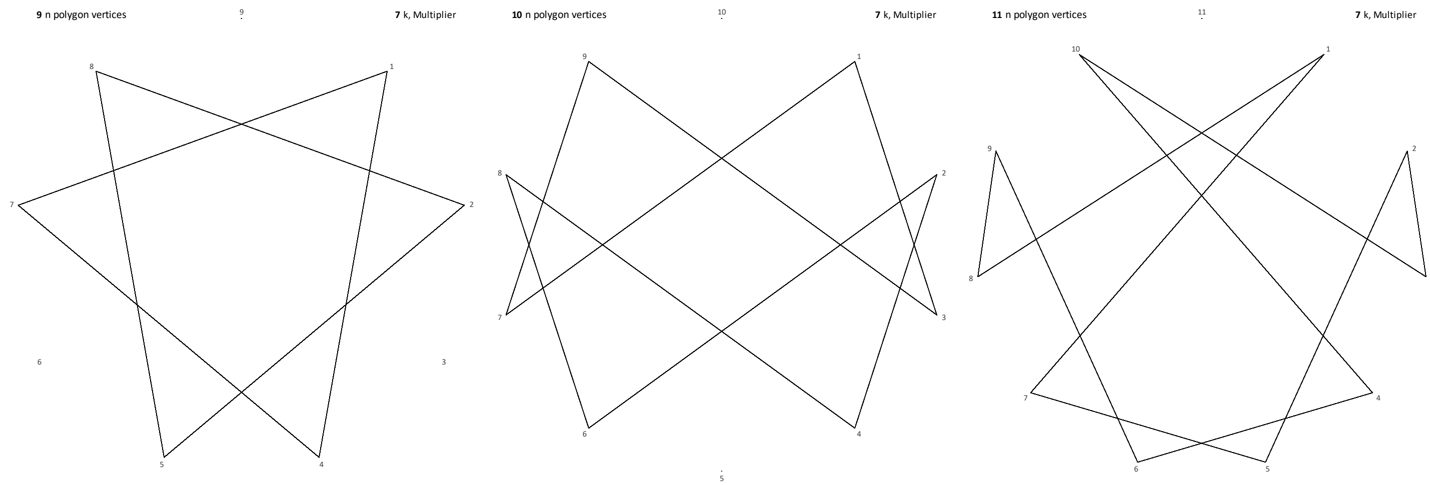
Degree of Rotational Symmetry

As defined in explainer 2.5b, an images' ***degree of rotational symmetry*** is *the number of different vertices it can be rotated to for which it matches the original.* In the context of cardioids, the degree of rotational symmetry is the greatest common divisor between n and k-1. These three examples all have k = 7 for n = 9, 10, and 11 from left to right.



The degrees of rotational symmetry are 3, 2, and 1 because GCD(9, 6) **=** 3, GCD(10, 6) = 2, and GCD(11, 6) = 1.

The left panel can be rotated 1/3 or 2/3 of a turn so that 3 or 6 replace the top vertex 9 without changing the image.

The middle image can be rotated ½ a turn so that 5 replaces 10 at the top without changing the image.

The right panel cannot be rotated without altering the image because, when there is no common factor between n and k-1, there is no rotational symmetry.

Note that all images maintain vertical symmetry for reasons discussed in explainer 11.2a. Further, if the degree of rotational symmetry is even (like the middle panel), then the image will have horizontal as well as vertical symmetry.

Why it works. Consider situations having a common divisor greater than 1. Let $G > 1$ be the largest common factor between n and k-1, G = GCD(n, k-1). Then $n = m \cdot G$ and $k-1 = h \cdot G$ where m and h are positive whole numbers.

Consider connecting vertices in this instance. Let j be a positive number between 1 and m. Consider how the starting points j and $m+j$ are connected to their ending points upon multiplication by k. Start by looking at $j = 1$.

Vertex 1 connects to k.

Vertex m+1 connects to $(m+1)k = mk + k = m(k-1) + [m+k] = m(hG) + [m+k] = h(mG) + [m+k] = hn + [m+k]$.

Vertex k differs from 1 by k-1 vertices. But m+1 connects with its ending vertex the same number of vertices after m+1 as 1 connects to k once h whole number rotations around the circle (n vertices) are removed.

Vertex j connects to jk.

Vertex $m+j$ connects to $(m+j)k = mk + jk = m(k-1) + [m+jk] = m(hG) + [m+jk] = hn + [m+jk]$.

Once again, starting and ending vertices differ by the same amount, once common values of n are removed. Put another way, if you rotated the image by m vertices, the image would look the same.

If $G > 2$, the same argument can be performed comparing starting at j and $2m+j$ and so on. In each instance, the next set of lines has this same rotational symmetry.

A caveat. If $n = 4j+2$ and $k = n/2+1$ the image (where each opposing vertex is connected to its counterpart half-way around the image) appears to have rotational symmetry of degree n. The smallest such image is n = 6 and k = 4. Vertex 1 connects to 4 but vertex 4 also connects to vertex 4 (4·4 = 16 = 2·6 + 4) so that the symmetry is an optical illusion. The calculated degree of symmetry remains n/2 (or 3 in this example) if we base that symmetry on HOW the lines were drawn. Checking $n = 4j$ and $k = n/2+1$ produces no such optical illusion as n = 8, k = 5 will confirm.