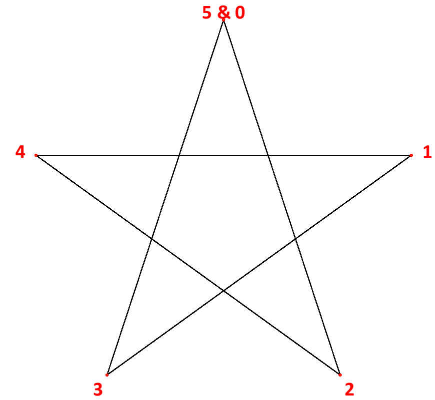
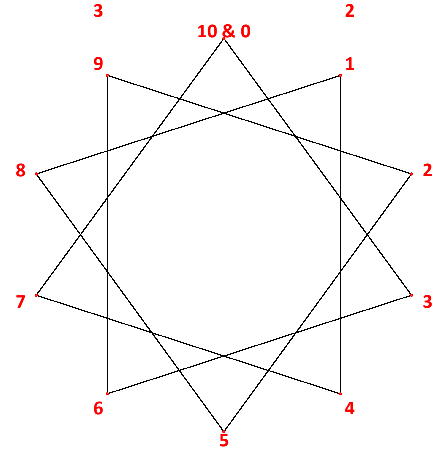
About *Relatively Prime* (or *Coprime*) Numbers

Many of the files in **PwP** create images which are based on *continuously drawn lines.* An image is continuously drawn if you start at a vertex (the top in ***PwP***) and add a line by following a rule for finding the next endpoint then draw the line and repeat this process until the endpoint of the last line drawn is the starting point of the first line drawn. The simplest example is the pentagram shown to the right.



There are two ways to *continuously draw* a five point star starting at the top: Either draw a line every 2nd vertex (from 0-2-4-1-3-0); or draw a line every 3rd (from 0-3-1-4-2-0). Both rules produce the same star, the only difference is whether that star appears drawn in a clockwise or counterclockwise fashion.

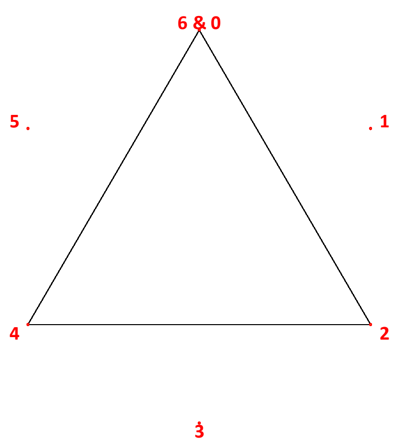
Similarly, we can continuously draw a 10-point star such as the one to the right by counting every 3rd or 7th vertex because 3 and 7 are *relatively prime* to 10.



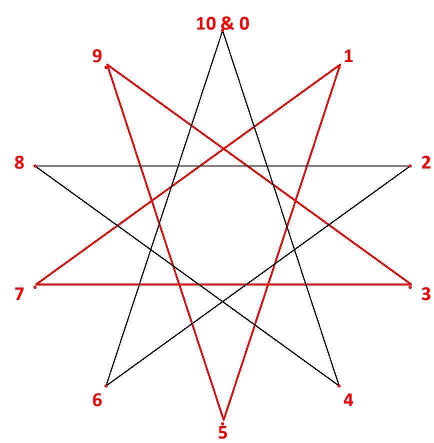
Definition: Two numbers ***a*** and ***b*** are ***relatively prime*** if their greatest common divisor is 1, GCD(***a***, ***b***) = 1. We also say that the numbers are ***coprime***.

But if we count every 4th line given ***n*** = 10, we end up with a pentagram like above (except the vertices are 0-4-8-2-6-0, this is the pentagram noted in black in the third image). Notice that only even vertices have been used in this instance. This is because ***n*** = 10 and ***J*** = 4 are not relatively prime, they contain a common factor of 2.

In this situation, it is not possible to draw a 10 point star with 4 vertex jumps. Of course, such stars exist but they are based on two circuits, one even and the other odd. Both circuits are provided in the third image to the right: **the black star shows even**; **the red star shows odd**. It is simply impossible to create this star using the "*Add 4 and draw a line*" rule because that rule will never move between odds and evens.



This is the same reason the ***n*** = 6, and ***J*** = 2 image to the left is a triangle rather than a six-pointed star. The "*Add 2 and draw a line*" rule that drew the pentagram at the top (when ***n*** = 5) produces a triangle here because this rule only includes even used vertices. This is because ***n*** = 6 and ***J*** = 2 are not relatively prime, they contain a common factor of 2. To produce the "other" triangle in the hexagram, one would need to start at an odd numbered vertex, just like the red star in the ***n*** = 10 and ***J*** = 4 image to the right.



This is not to say that ***J*** = 2 uses only even vertices. It is easy to see that odd vertices are used in the second pass around the circle if ***n*** is an odd number. Consider the ***n*** = 9 and ***J*** = 2 image. The used vertex order is 0-2-4-6-8-1-3-5-7-0. The first half are even, the second half are odd (the final can be viewed as both even and odd if one notes that it is both ***n*** = 9 and 0). The same pattern is seen in the ***n*** = 5, ***J*** = 2 pentagram we started with above.

A general rule. To create an ***n***-point star one need only find a number ***J*** with 1 < ***J*** < ***n***-1 such that ***n*** and ***J*** have no factors in common. Such ***J*** are relatively prime to ***n*** (or coprime) and have GCD(***J***, ***n***) = 1. *CLAIM*: This is possible for every ***n*** > 4 with the exception of ***n*** = 6. One can always create an ***n***-sided polygon because ***J*** = 1 and ***J*** = ***n***-1 are both coprime to ***n***.

Finding the GCD of two numbers. When numbers are "small" it is relatively easy to find the greatest common divisor of two numbers because all that needs to be done is to compare the prime factorizations of both numbers. But when the numbers are larger and such factorization becomes tedious, we can be happy that more than 2000 years ago, Euclid provided us with a very nice algorithm for finding the GCD of two numbers (see the Euclid's Algorithm *explainer*).