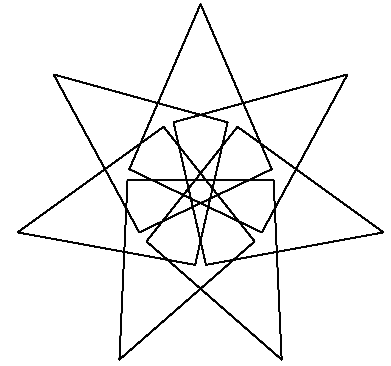# An Analytical Approach to Line Placement

The images to the left and right are based on common $n = 7$, $S = 3$, and $P = 5$. Each image has 21 line segments. The only difference is $J = 2$ to the left and $J = 3$ to the right. These examples will be used to explain line placement in the final image. Any line segment can be fully described by noting the starting point and ending point of the segment.

Number and location of endpoints, B. Each line segment endpoint in the final image must be a part of the Vertex Frame, VF. In general, there are $B = nS/VCF$ **possible subdivisions** in the final image (VCF = GCD(n,J)). If VCF > 1 we obtain the same image by dividing both n and J by VCF, therefore *we restrict our discussion to n and J that are relatively prime and $B = nS$.* (For example, the VF for $n = 10$, $J = 4$ (VCF = 2) is the same as $n = 5$, $J = 2$; both VFs are clockwise drawn pentagrams ($n = 5$, $J = 3$ is counterclockwise drawn).)

How many endpoints are used? If SCF = 1 (SCF = GCD($nS/VCF,P$)), then all B subdivision points are used in the final image. The order in which these points are used determines the ultimate image.

When SCF > 1, a total of B/SCF subdivision endpoints are used in the final image. (For $n = 7$, $S = 3$, all 21 possible endpoints are used if $P = 5$; 7 are used if $P = 6$ (since SCF = 3) and 3 are used when $P = 7$ (since SCF = 7).) When SCF > 1, each endpoint is a multiple of SCF (once modular considerations are controlled for). To make this explicit, consider $P = 6$. The 7 counted subdivisions used to complete the image in this instance are 6, 12, 18, 24, 30, 36, and 42 as shown in the table to the right but the last four of these values are counted in the second time around the VF at endpoints 3 = 24 MOD 21, 9 = 30 MOD 21, 15 = 36 MOD 21 and 0 = 42 MOD 21. In the end, all subdivision endpoints used to create the final image are multiples of 3 = SCF. By contrast, the 3 subdivisions that are used when $P = 7$ are easy to see: 7, 14, and 0 = 21 MOD 21.

Subdivision endpoint		
k,	kP,	$E_k = $ MOD(kP,B)
1	6	6
2	12	12
3	18	18
4	24	3
5	30	9
6	36	15
7	42	0

Which endpoints are used? The first line starts at the top of the n-gon and ends after P subdivisions have been counted along the VF, the second line goes from this point to the endpoint that is $2P$ subdivisions along the VF, and so on. This continues until the final endpoint connects with the initial starting point at the top of the n-gon. Note that this counting occurs even if $kP > B$ (as we saw in the example above). The B/SCF used endpoints are $E_k = $ MOD(kP,B) for $1 \le k \le B$/SCF.

Where on the VF is a used endpoint located? Each used endpoint is kP subdivisions along the VF. The location of that point depends on both S, the number of subdivisions per segment of the VF, as well as n and J. The location can be thought of as a two-part process: **I)** Find which part of the VF the point is on; and **II)** Find where that segment of the VF is located based on polygonal vertices. An example shows the relevant calculations based on four parameters in yellow.

I) Let $C = kP$ be the counted number of endpoints that is the location of a used subdivision of the final image. If we divide C by S, we obtain a number that has a whole part, W, and a fractional part, r/S, where $r < S$. W tells us the starting point of the VF segment on which this point is located, and r tells us how far towards the next segment the endpoint is located.

II) The polygon vertex for VF_{start} and VF_{end} is based on n, J, and W. Each VF vertex is J vertices of the n-gon past the previous vertex. Equations for start and end vertices are as noted in the last 2 lines of the example to the right. The next page shows the how the above images are created using the same calculations.

372	$C = kP$, counted endpoint
17	S
21 15/17	C/S written as a mixed number, W r/S
	The endpoint is on the $W+1^{st}$ segment of the VF from W to $W+1$. Each segment of the VF has S subdivisions. The endpoint is r subdivisions along this segment of the VF from W to $W+1$. (In Excel, $W = $ INT(C/S) and $r = $ MOD(C,S).)
	The VF segment location relative to polygon vertices 0 - n-1 is based on n & J. n-gon vertices for this segment are in red.
13	n
6	J
9	Location of W on VF, ($VF_{start} = $ MOD(WJ,n))
2	Location of $W+1$ on VF ($VF_{end} = $ MOD($VF_{start}+J,n$))

Both images are based on a 7-gon with **S** = 3 and **P** = 5. The images show the **7 vertices in red**, **vertex frame in blue**, and **possible subdivisions in green**. The only difference is **J** = 2 to the left and **J** = 3 to the right. The upper table shows the order of subdivision usage, and the lower shows location of each point on the VF.
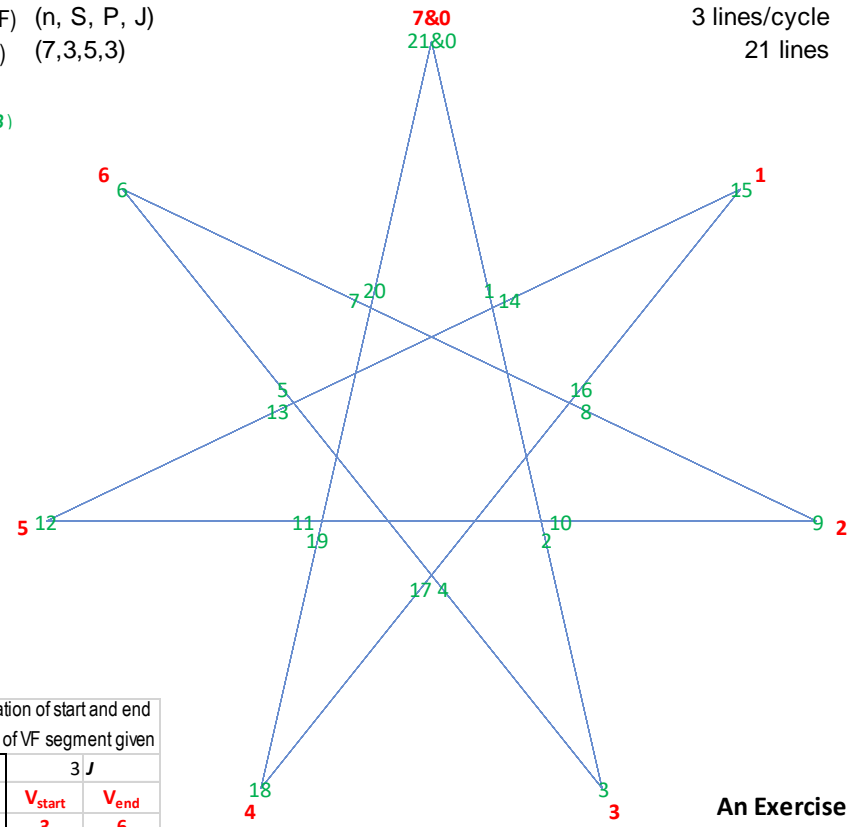
(n, S, P, J)
(7,3,5,2)

7&0
21&0

3 lines/cycle
21 lines

(n, S, P, J)
(7,3,5,3)

7&0
21&0
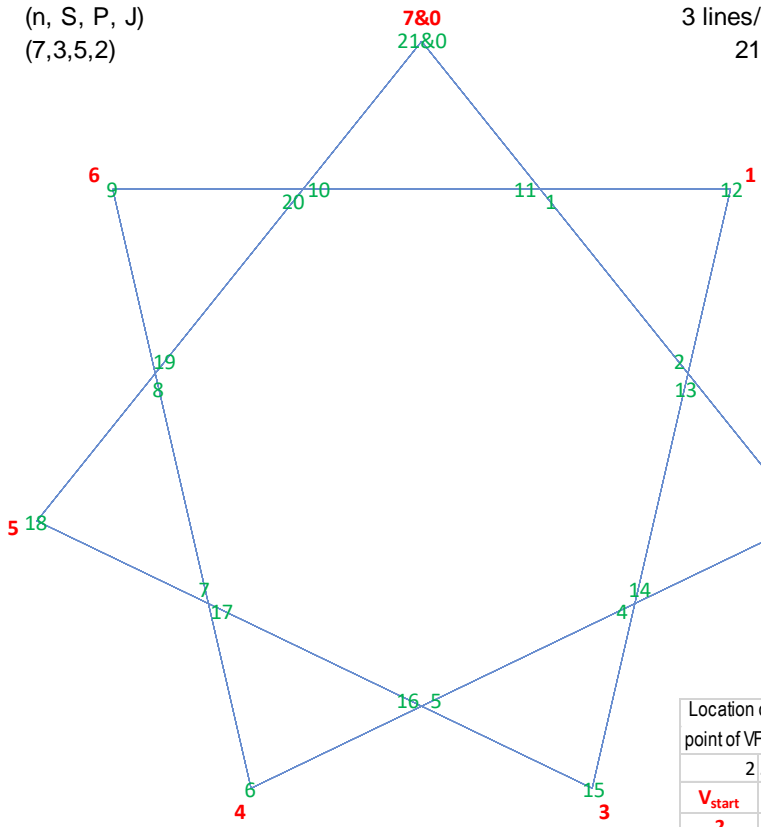
3 lines/cycle
21 lines

Subdivision Endpoints
(MIN(50,$B=nS$/VCF)
E_k endpts listed.)
Subdivision
k, kP, $E_k =MOD(kP,B)$

k	kP	E_k
1	5	5
2	10	10
3	15	15
4	20	20
5	25	4
6	30	9
7	35	14
8	40	19
9	45	3
10	50	8
11	55	13
12	60	18
13	65	2
14	70	7
15	75	12
16	80	17
17	85	1
18	90	6
19	95	11
20	100	16
21	105	0

Location of start and end point of VF segment given				7 n		Location of start and end point of VF segment given	
				3 S			
2 J				5 P		3 J	
V_{start}	V_{end}	k	C=kP	W	r	V_{start}	V_{end}
2	4	1	5	1	2	3	6
6	1	2	10	3	1	2	5
3	5	3	15	5	0	1	4
5	0	4	20	6	2	4	0
2	4	5	25	8	1	3	6
6	1	6	30	10	0	2	5
1	3	7	35	11	2	5	1
5	0	8	40	13	1	4	0
2	4	9	45	15	0	3	6
4	6	10	50	16	2	6	2
1	3	11	55	18	1	5	1
5	0	12	60	20	0	4	0
0	2	13	65	21	2	0	3
4	6	14	70	23	1	6	2
1	3	15	75	25	0	5	1
3	5	16	80	26	2	1	4
0	2	17	85	28	1	0	3
4	6	18	90	30	0	6	2
6	1	19	95	31	2	2	5
3	5	20	100	33	1	1	4
0	2	21	105	35	0	0	3

A Final Note

A cycle is completed once **r** = 0.

Both of these cycles are 3 lines long.

Both image's first cycle is shown in red on the images to the right.
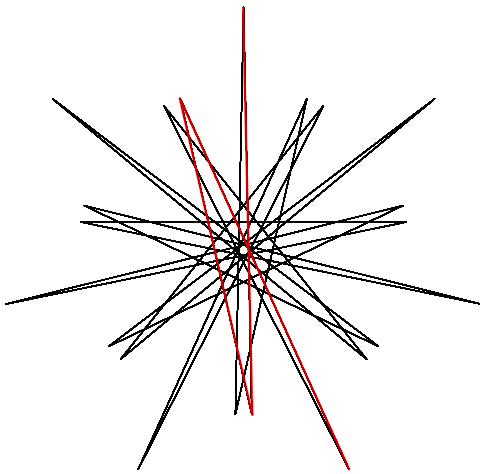
An Exercise

Use the numbered images above to sketch out what the images would look like if **P** = 2 rather than 5. Check your work using the file. How do these images compare to the triangular one in Explainer 2.1?