Analyzing Patterns in Continuously Drawn *n,J*-Stars

One interesting result from the [Drawing the VF](#) explainer is that many continuously drawn ***n,J***-stars (which forms the VF) are filled in sequentially (on either the 1 to ***J***-1 or ***n***-1 to ***n***-***J***+1 side). This explainer dives more deeply into the conditions under which this might occur (and when it does not occur).

We noticed there that the order of filling in vertices was symmetric about ***n***/2 so we can (and here will) restrict our discussion to ***J*** < ***n***/2. As a result, this explainer reformats the tabular information from the earlier explainer to focus on ***J*** < ***n***/2 which produce ***n***-point stars. (When ***n*** is prime, all ***J*** from 2 to INTEGER(***n***/2) produce distinct continuously drawn stars but when ***n*** is a composite number, we require ***n*** and ***J*** to be coprime (VCF = GCD(***n***,***J***) = 1) in order to create a continuously drawn ***n***,***J***-star (as is discussed in explainer [1.2](#)). The angles created by such star points are discussed in [1.5](#).)

n		Times around ***ending vertex, T*** (***T*** < ***J***)							
19	***J*** \ ***T***	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
	2	1							
	3	2	1						
	4	1	2	3					
	5	1	2	3	4				
	6	5	4	3	2	1			
	7	2	4	6	1	3	5		
	8	5	2	7	4	1	6	3	
	9	8	7	6	5	4	3	2	1

The table to right provides the same information but in a more compact format. The location of vertex 1 is automatically highlighted in each row in the Sequential Polygons and Stars *Excel* file which allows analysis up to ***n*** = 66. If there is a 1 in the ***T*** = 1 column, then the VF fills in from 1 to ***J***-1 ↻ (clockwise) and when 1 is the final entry in the row (such as ***J*** = 3, 6, and 9) then the VF fills in ↺ (counterclockwise) from ***J***-1 to 1. (The 1 in the ***J*** = 2 row is both since it is a one entry series.)

Claim: ***J*** = 2, 3, and 4 are **always** sequential (assuming VCF = 1). The first two are straightforward. **A)** If ***J*** = 2 then ***n*** must be odd. The VF endpoints are 2, 4, … ***n***-1, 1, 3, … , ***n***-2, ***n***&0 with the first time around using even vertices and the second time around with odd vertices. (Of course, a sequence involving only one value is tautologically a sequence, albeit an uninteresting one.) **B)** If ***J*** = 3, then ***n*** is of the form ***n*** = 3***k***+1 or ***n*** = 3***k***+2. The former (like ***n*** = 19 above) produces an endpoint of 2 for ***T*** = 1 so the next time around must have remainder of 1 = 2+2 MOD 3. The latter (like ***n*** = 20) would have an endpoint of 1 for ***T*** = 1 and therefore 2 for ***T*** = 2. Both possibilities are sequential.

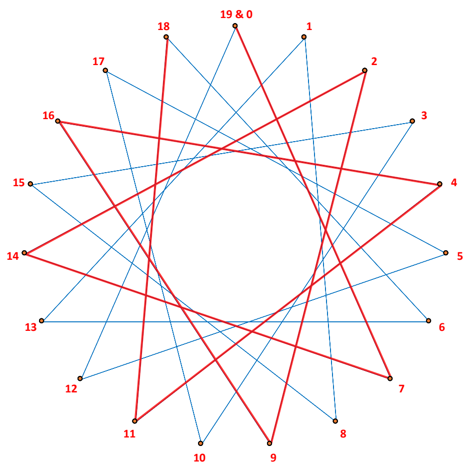
C) The ***J*** = 4 claim is a bit more interesting. To produce a continuously drawn ***n***-star with ***J*** = 4, ***n*** must be odd. The first VF crossing, ***T*** = 1, must go from a multiple of 4 to the next multiple of 4 but any multiple of 4 is a multiple of 2 so the starting point for the crossing line must be at **even** vertex ***n***-1 or ***n***-3 in which case the ending vertex is at 3 or 1. The second VF crossing vertex is 2 in both instances (2 = 3+3 MOD 4 in the first instance and 2 = 1+1 MOD 4 in the second). The third VF crossing completes the sequence: 1 = 2+3 MOD 4 in the first and 3 = 2+1 MOD 4 in the second.

The *T* = 1 endpoint of the VF tells the whole story. Call this endpoint ***E*** (so ***E*** = 2 with ***J*** = 7 or ***E*** = 5 with ***J*** = 8 in the table above). The order of subsequent fill-in vertices is ***kE*** MOD ***J*** for 2 ≤ ***k*** ≤ ***J***-1.

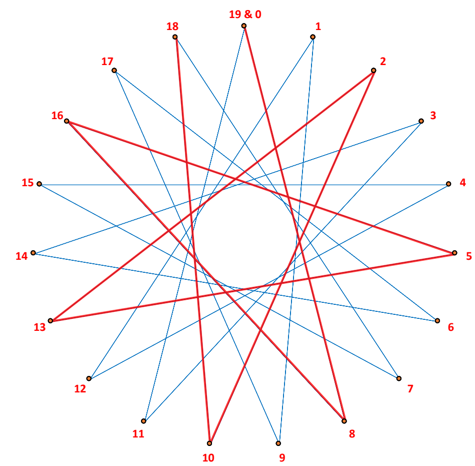
To understand *n*, look at factors of *n*-1 and *n*+1 less than n/2. If you want to have ***E*** = 1 (for ***T*** = 1) it must be the case that ***n***+1 is a multiple of ***J***. This would produce an increasing sequence that fills in endpoints ↻ from 1 to ***J***-1; the ***J***-1 lines that form the internal lines for the star that intersect the vertical radius starting at point ***n***&0. In the above example, ***n***+1 = 20 has factors 2, 4, and 5 < ***n***/2). This implies that ***J*** = 2, 4, and 5 have sequences that increase from 1 to ***J***-1. By contrast, ***n***-1 = 18 has factors 2, 3, 6, and 9 < ***n***/2). This means that ***J*** = 2, 3, 6, and 9 have sequences that decrease ↺ from ***J***-1 to 1. (***J*** = 2 is on both lists, as noted above.)

n/J* provides information on the VF as Rotating Polygons.** As noted above, ***J =2, 3, and 4 always produce sequential images (which are therefore viewed as versions of rotating polygons discussed in [1.4](#)). The table to the right provides ***n/J*** whole number values for ***J*** = 2 through 4, and closest mixed fraction values to a single digit for ***J*** > 4 (given ***n*** = 19). The type of the rotating polygon is given by the closest number to ***n/J*** if the sequence of ending points **starts or ends** with 1 so ***J*** = 5 is a rotating quadrangle, ***J*** = 6 is a rotating triangle, and ***J*** = 19 is a rotating line. Additionally, if the fractional part is just below 1, the rotation is ↻ (like ***J*** = 4 and 5); and if it is just above 0, the rotation is ↺ (like ***J*** = 3, 6, and 9).

n /***J*** (1 digit fraction)	***n*** 19	***J*** \ ***T***
10		**2**
6		**3**
5		**4**
3 4/5		**5**
3 1/6		**6**
2 5/7		**7**
2 3/8		**8**
2 1/9		**9**



Rotating Stars. When the vertex frame lines are not filled in in a sequential fashion, one can still obtain interesting insights from ***n*/*J***. Note from the top table above, ***J*** = 7 and ***J*** = 8 both have 1 in the third cell from the end of the sequence (at column ***T*** = ***J***-3). Both images are ↻ rotating 3-jump stars. The ***J*** = 7 version is shown to the left, and the ***J*** = 8 version is shown to the right. The fraction ***n*/*J*** can be used to determine what type of star is appropriate in both situations.



Left image: The whole number portion of 19/7 is 2 and the fractional portion is 5/7. 5/7 is closest to 2/3 and we know we want a 3-jump star. The number of points in the rotating star is the whole portion times 3 plus the numerator of the fraction 2/3 (or 8 = 2·3+2). Since the ending 1 was 3 from the right (at ***T*** = 4), the star is open at the top and rotates ↺.

Right image: The whole number portion of 19/8 is 2 and the fractional portion is 3/8. 3/8 is closest to 1/3 and we know we want a 3-jump star. The number of points in the rotating star is the whole portion times 3 plus the numerator of the fraction 1/3 (or 7 = 2·3+1). Since the ending 1 was 3 from the right (at ***T*** = 5), the star is open at the top and rotates ↺.

To finish this discussion of stars as rotating polygons or stars it is worthwhile to remember that the underlying VF is a star such as the 19,7-star to the left or the 19,8-star to the right. But each of these stars also can be thought of as being created in a sequential fashion from rotating smaller-sized stars (a ↺ 8,3-star of the left and a ↺ 7,3-star on the right).

n*/*J (1 digit	***n***		Times around ***ending vertex***, ***T*** (***T*** < ***J***)																											
fraction)	**60**	***J*** \ ***T***	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**
30		**2**																												
20		**3**																												
15		**4**																												
12		**5**																												
10		**6**																												
8 4/7		**7**	3	6	2	5	1	4																						
7 1/2		**8**																												
6 2/3		**9**																												
6		**10**																												
5 4/9		**11**	6	1	7	2	8	3	9	4	10	5																		
5		**12**																												
4 3/5		**13**	5	10	2	7	12	4	9	1	6	11	3	8																
4 2/7		**14**																												
4		**15**																												
3 3/4		**16**																												
3 1/2		**17**	8	16	7	15	6	14	5	13	4	12	3	11	2	10	1	9												
3 1/3		**18**																												
3 1/6		**19**	16	13	10	7	4	1	17	14	11	8	5	2	18	15	12	9	6	3										
3		**20**																												
2 6/7		**21**																												
2 5/7		**22**																												
2 3/5		**23**	9	18	4	13	22	8	17	3	12	21	7	16	2	11	20	6	15	1	10	19	5	14						
2 1/2		**24**																												
2 2/5		**25**																												
2 1/3		**26**																												
2 2/9		**27**																												
2 1/7		**28**																												
2		**29**	27	25	23	21	19	17	15	13	11	9	7	5	3	1	28	26	24	22	20	18	16	14	12	10	8	6	4	2

With this in mind, 6 of the 7 60-point VF stars noted above can be recast as rotating stars. ***J*** = 7 is ↺ 17,2; ***J*** = 11 is ↻ 11,2; ***J*** = 13 is ↺ 23,5; ***J*** = 17 is ↺ 7,2; ***J*** = 19 is ↻ 19,6; and ***J*** = 23 is ↺ 13,5. Verify each dynamically [at this link]() using the **Single Line Drawing** mode by setting **Drawn Lines** to the number of points in the rotating star (e.g., 13 for ***J*** = 23).

Two additional patterns emerge in the ***n*** = 60 table. First, the 1 is right in the middle of the ***J*** = 29 row and 2 is the last entry (for ***T*** = 28). Indeed, the numbers decline with all odd values, then all even values. The 2 in the final position means that this is a 2 time around ↺ line (60/29 is close to but a bit larger than 2) so set **Drawn Lines** = 2, ***J*** = 29 and watch the VF get filled in using the odd first, then the even vertices on the right half of the 60-gon.

Twin Primes. The second point to note given ***n*** = 60 is that there are NO sequential values for ***J*** in this instance. This occurs for an interesting reason: 60 is sandwiched between two primes, 59 and 61. Such pairs of primes are called *twin primes*. Whenever this occurs, there are no factors (other than 1 and the number) of the adjacent numbers. But above we saw that factoring the numbers just above and just below ***n*** tell us what ***J*** values produce sequentially drawn stars. This is not unique to ***n*** = 60, the same pattern occurs for ***n*** = 12, 18, 30, and 42, because each is surrounded by primes.

Additional patterns based on primes. The table below, reproduced from the Sequential Polygons and Stars *Excel* file summarizes polygon and star information for 5 ≤ ***n*** ≤ 66. The prime ***n*** values are highlighted for convenience. In addition to the absence of sequentially drawn polygons when ***n*** is between a twin prime, you can see that there are additional patterns based on whether ***n*** is at or near a prime.

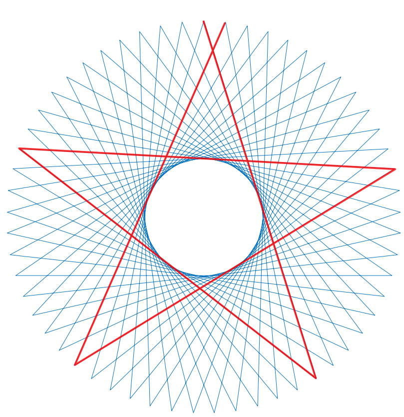
If ***n*** is prime, there are INTEGER(***n***/2) – 1 total possibilities since all 2 ≤ ***J*** ≤ INTEGER(***n***/2) have GCD(***n***,***J***) = 1. Any composite ***n*** has fewer possibilities since when ***n*** is composite (than nearby primes) because some of the ***J*** have factors in common with ***n***. One cannot create a continuously drawn ***n***,***J*** star if GCD(***n***,***J***) > 1.

If ***n*** is one more than a prime (i.e., ***n***-1 is prime), then any sequentially rotating polygon must be based on a factor of ***n***+1. As a result, the cross-over endpoint for the first line is at ***E*** = 1. As noted in the table, each increases (rotates ↻).

If ***n*** is one less than a prime (i.e., ***n***+1 is prime), then any sequentially rotating polygon must be based on a factor of ***n***-1. As a result, the start of the first cross-over line is ***n***-1 and the endpoint is ***E*** = ***J***-1. Many examples are noted in the table (as *all decreasing, 1 less than prime*), each rotates ↺. Note that ***n*** in each instance is even (***n***+1 is prime).

There are additional odd ***n*** (13, 21, 25, 33, 37, 45, 57, and 61) for which all sequential ***J*** are decreasing (except for ***J*** = 2 which, as noted above, is both). Clearly ***n*** is not one less than a prime in this case (since ***n***+1 is even and therefore divisible by 2). The unifying attribute of each of these numbers is that in each instance, there are only two non-trivial factors of the number: 2 and (***n***+1)/2. Put another way, (***n***+1)/2 is prime. Since (***n***+1)/2 > ***n***/2, it violates our restriction on ***J*** that ***J*** < ***n***/2.

An Exercise. The 57,***J***-star below can quite clearly be described as a ↻ rotating 5,2-star. ***Without counting vertices,*** what is the value of ***J*** here?



	Sequential?		Patterns of Rotating Polygons and Stars		Sequential?		
n	# yes	# no	Notes on sequential polygons	***n***	# yes	# no	Notes on sequential polygons
5	1			36	2	3	all decreasing, 1 less than prime
6			no continuous 6-point star	37	7	10	all decreasing (except 2)
7	2			38	2	6	all increasing, 1 more than prime
8	1			39	6	5	
9	2			40	2	5	all decreasing, 1 less than prime
10	1			41	10	9	
11	4			42		5	Surrounded by twin primes
12		1	Surrounded by twin primes	43	8	12	
13	4	1	all decreasing (except 2)	44	4	5	all increasing, 1 more than prime
14	2		all increasing, 1 more than prime	45	4	7	all decreasing (except 2)
15	3			46	4	6	all decreasing, 1 less than prime
16	2	1	all decreasing, 1 less than prime	47	8	14	
17	5	2		48	1	6	7 sequential n=7^2-1 & 47 prime
18		2	Surrounded by twin primes	49	10	10	
19	6	2		50	3	6	
20	2	1	all increasing, 1 more than prime	51	6	9	
21	4	1	all decreasing (except 2)	52	2	9	all decreasing, 1 less than prime
22	2	2	all decreasing, 1 less than prime	53	8	17	
23	6	4		54	2	6	all increasing, 1 more than prime
24	1	2	5 sequential, n = 5^2-1 & 23 prime	55	10	9	
25	6	3	all decreasing (except 2)	56	4	7	
26	3	2		57	6	11	all decreasing (except 2)
27	4	4		58	2	11	all decreasing, 1 less than prime
28	2	3	all decreasing, 1 less than prime	59	10	18	
29	8	5		60		7	Surrounded by twin primes
30		3	Surrounded by twin primes	61	10	19	all decreasing (except 2)
31	8	6		62	4	10	all increasing, 1 more than prime
32	2	5	all increasing, 1 more than prime	63	5	12	
33	4	5	all decreasing (except 2)	64	6	9	
34	4	3		65	9	14	
35	7	4		66	2	7	all decreasing, 1 less than prime