# Modular Analysis of *Two Footballs* Cycles

The Two Footballs explainer proposed the following rule for creating *two footballs* images:

Base *two footballs* images off of $J$. These images occur when $n = P = 3J \pm 1$ and $S = 2J \pm 2$.

The current explainer examines why there are 1, 2, or 4 cycles in a given *two footballs* image.

**Single cycle images.** In that explainer, the claim was made that if $n$ is odd there must be a single cycle. When will $n$ be odd? Since $n = P = 3J \pm 1$, $n$ will be odd if $3J$ is even. Since 3 is odd, $3J$ is even only if $J$ is even. This means that ***two footballs images will be single cycle whenever J is even.***

**Multiple cycle images.** Multiple cycle *two footballs* images can only occur if $J$ is odd.

For each $J$ there are two $S$ and $n = P$ values that produce *two footballs* images, one version uses plus signs the other uses minus signs in the equations for $n = P$ and $S$.

**CLAIM:** For each odd $J$, one version produces a 2-cycle image and the other produces a 4-cycle image.

First, consider $S$. Recall, $S = 2J \pm 2$. If $J$ is odd, then $2 = 2J$ MOD 4 or $2J$ is divisible by 2 but not 4.

Consider the adding 2 version of $S$: $S = 2J + 2$ so that $S = 2J + 2 = 2 + 2 = 4 = 0$ MOD 4.
The subtracting 2 version of $S$ is even faster: $S = 2J - 2$ so that $S = 2J - 2 = 2 - 2 = 0$ MOD 4.
*In both instances, if J is odd, S is divisible by 4.*

The difference between 2-cycle and 4-cycle versions depends on $n = P$ when $J$ is odd.

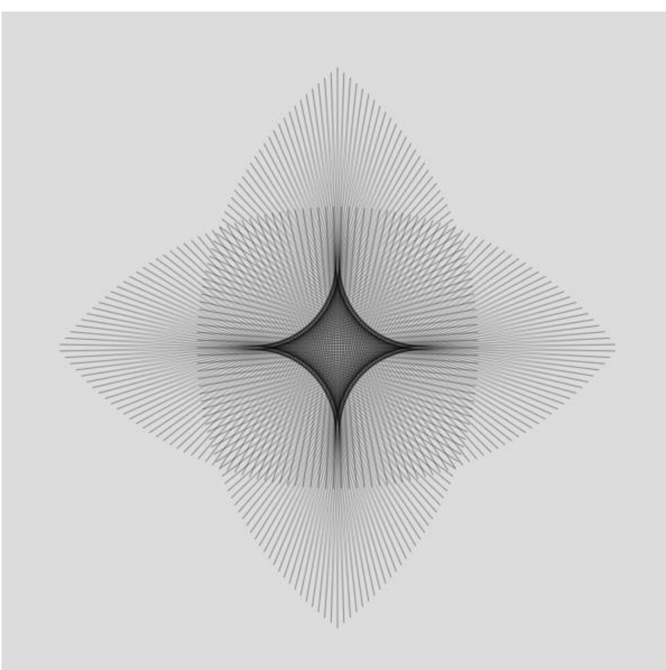
Consider $n = P$ for odd $J$. Recall $n = P = 3J \pm 1$. If $J$ is odd, it must be of the form $J = 4k+1$ ($J = 1$ MOD 4) or $J = 4k+3$ ($J = 3$ MOD 4). We consider these in turn.

**$J = 1$ MOD 4:** Consider $n = P = 3J + 1$. $3J = 3 \cdot 1 = 3$ MOD 4 so that $n = P = 3J + 1 = 3 + 1 = 4 = 0$ MOD 4.
*The "+1" version has 4-cycles given J = 1 MOD 4.*

Consider $n = P = 3J - 1$. $3J = 3 \cdot 1 = 3$ MOD 4 so that $n = P = 3J - 1 = 3 - 1 = 2$ MOD 4.
*The "–1" version has 2-cycles given J = 1 MOD 4.*

**$J = 3$ MOD 4:** Consider $n = P = 3J + 1$. $3J = 3 \cdot 3 = 9 = 1$ MOD 4 so that $n = P = 3J + 1 = 1 + 1 = 2$ MOD 4.
*The "+1" version has 2-cycles given J = 3 MOD 4.*

Consider $n = P = 3J - 1$. $3J = 3 \cdot 3 = 9 = 1$ MOD 4 so that $n = P = 3J - 1 = 1 - 1 = 0$ MOD 4.
*The "–1" version has 4-cycles given J = 3 MOD 4.*



We now see how these rules apply to the images in the Two Footballs explainer.

The top image has $S = 166$ lines, $n = P = 247$ and $J = 82$. It has 1-cycle since $J$ is even.

The middle image has $S = 168$ lines, $n = P = 250$ and $J = 83$. $J = 3$ MOD 4 and $n = P = 3J + 1$ so this is a 2-cycle image (3rd option above).

The bottom image has $S = 164$ lines, $n = P = 248$ and $J = 83$. $J = 3$ MOD 4 and $n = P = 3J - 1$ so this is a 4-cycle image (4th option above).

**An example.** The image to the left is dense enough that it may be difficult to see how many cycles it has. It is an $n = P = 490$, $S = 328$, $J = 163$ image. How many cycles does it have and why?