Analyzing Stars inside of a Star

The Stars inside of a Star explainer had several assertions regarding patterns of line intersections. This explainer examines why those assertions are correct. We also consider generalizations available based on these observations.

Why are there 2 intersections per jumped over vertex? An ***n*****,*****J***-star (***J*** < ***n***/2) has ***n*** lines. This does not mean that there is one line per vertex because each line uses two vertices, one denotes where the line starts, the other where it ends. Each of the ***J***-1 vertices from **1** to ***J*****-1** have "other endpoints" that MUST be vertices from ***J*****+1** to ***n*****-1** since otherwise the difference between vertices is less than ***J***. With ***J*** < ***n***/2, those lines with starting point at vertices **1** to ***J***-1 are to the right and below the center while those ending at these vertices started on lines which are above the center.

Why do the interior angles created by these intersections match other *n*-stars? As noted in the *Angles in Stars and Polygons* explainer, the angle is α = 180(***n***-2***J***)/***n*** using the *Inscribed Angle Theorem*. For *Sharpest Stars*, this simplifies to 180/***n*** if ***n*** is odd and 360/***n*** if ***n*** is even. These are the values shown in the top row of the table below (associated with ***J*** = 5 for ***n*** = 11 and 12, annotated below and in images at left and right).

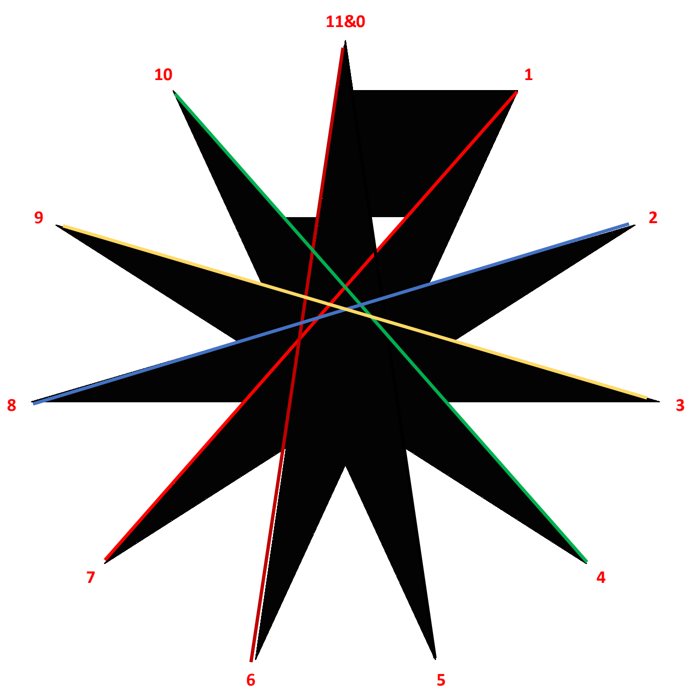
Consider how the ***n*****,*****J*** angle relates to the ***n*****,*****J***-1 angle using the angle equation above. Replacing ***J*** with ***J***-1 in the above equation increases the numerator by two vertices ((***n***-2(***J***-1)) = (***n***-2***J***+2)).

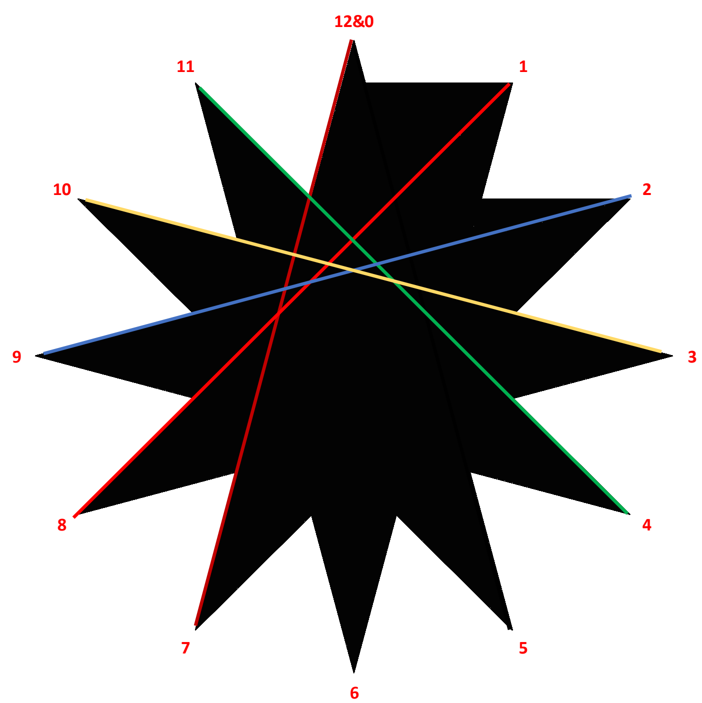
The *Interior Angle Theorem* generalizes the *Inscribed Angle Theorem* to allow for interior intersections. In this setting, the angle is the sum of vertices spanned on both sides of the intersection point under consideration times 180/***n***. (When the number of vertices spanned on one side is 0, the intersection is at a vertex just like at ***J*** = 5 to left and right.)

The table shows what happens as we move zigzag from intersections 5 to 4 to 3 to 2 to 1. Each move changes one line which increases spanned vertices by 1 at top and 1 at bottom.

Line spanning		Line spanning		Vertices spanned at		Angle°	Point	Line spanning		Line spanning		Vertices spanned at		Angle°
Vertices	Color	Vertices	Color	Top, T	Bottom, B	180(T+B)/***n***	label*	Vertices	Color	Vertices	Color	Top, T	Bottom, B	180(T+B)/***n***
0-6	**Brown**	**0-5**	**Black**	0	1	16.36	5	**0-7**	**Brown**	**0-5**	**Black**	0	2	30
1-7	**Red**	**0-5**	**Black**	1	2	49.09	4	**1-8**	**Red**	**0-5**	**Black**	1	3	60
1-7	**Red**	**4-10**	**Green**	2	3	81.82	3	**1-8**	**Red**	**4-11**	**Green**	2	4	90
2-8	**Blue**	**4-10**	**Green**	3	4	114.55	2	**2-9**	**Blue**	**4-11**	**Green**	3	5	120
2-8	**Blue**	**3-9**	**Gold**	4	5	147.27	1	**2-9**	**Blue**	**3-10**	**Gold**	4	6	150

*Point labels noted in *Stars inside a Star* explainer. Angle degree measurements are based on applying the *Interior Angle Theorem*.





We see that the move from internal intersection point ***J*** to ***J***-1 (5 to 4 or 4 to 3, etc.) increases the numerator by 2 vertices, just like changing from an ***n***,***J***-star to an ***n***,***J***-1 one using the *Inscribed Angle Theorem*. Both types of stars have the same angles. And, of course, the *Inscribed Angle Theorem* applies even if GCD(***n***,***J***) > 1 in which case one cannot create a continuously-drawn ***n***,***J***-star.

Why is there symmetry about the midpoint of the line segment from 0 to 5 in both images? If you look at the lines starting at vertex **1** and ***J*-1** you will see that the line (above the center) from ***n*-*J*+1** to **1** and the line from ***J*-1** to **2*J*-1** are the lines closest to the vertices at **0** and ***J***. Similarly, the interior lines from each of these vertices form the closest to the center intersections on line **0-5**. A similar argument applies for vertices **2** and ***J*-2**, etc.

Divisible Stars. The lines of a continuously-drawn ***n***,***J***-star create the initial star as well as ***J***-1 additional regular images. There are ***J***-2 stars and one polygon. The images can be described as ***n***,***k***-stars, 1 < ***k*** < ***J***, nested inside one another with the smallest image being an ***n***-gon. The interesting point is that this happens even if one cannot create a continuously-drawn ***n***,***k***-star because GCD(***n***,***k***) > 1.

To be explicit, if you set ***n*** = 12 and ***J*** = 4 you obtain an equilateral triangle connecting vertices **0**, **4** and **8**. As we see above, setting ***J*** = 5 produces 4 equilateral triangles, one of which has a point at the **red**-**black** line intersection noted as 4 in the blow-up above the right image on page 1. This peak is midway between vertex **0** and **1,** tilted 15° from the **0-4-8** one. More generally, all 12 equilateral triangle points (from the four equilateral triangles) are at the “half-hour,” midway between vertices of the 12-gon. These form the vertices of the internal 12,4-star.

One final point should be made about the equilateral triangles. If you focus on every other triangle, you end up with a 6,2 internally-drawn star. The pairs to consider are two hours apart: 12:30 and 2:30 or 1:30 and 3:30.

Right Angles. The intersection marked 3 in the right image is a right angle as noted in the table. It is one of 12 such intersections based on 3 squares. As compared with the equilateral triangles above, these 12 points are “on the hour” so the internal 12,3-star has a peak on the vertical centerline.

Zigzag Peaks. The internal stars and polygons peaks profiled in the table are on two radiuses, the vertical radius, and the radius midway between vertices **0** and **1**. (We could think of this as *on the hour* versus *on the half* (hour) if we think of the ***n***-gon as an ***n***-hour clock.) One should not conclude that odd internal star peaks must be on the vertical radius simply from the examples provided. Had we examined the internal stars based on a 13-6-star, the even peak points would be on the hour and the odd peak points would be on the half.

Parallel Lines. The 11,5 image has no parallel lines and each of the lines in the 12,5 has a parallel counterpart. This dichotomy is, indeed, general. Any ***n***-gram will have no parallel lines if ***n*** is odd and ***n***/2 pairs of parallel lines if ***n*** is even. The reason is instructive. Suppose you have an ***n***,***J***-star. If two lines are parallel, then they do not cross and therefore they span 2***J*** of the ***n*** vertices. Notice that 2***J*** is even regardless of ***J*** so that the remaining un-spanned ***n***-2***J*** vertices is even if ***n*** is even but odd if ***n*** is odd. Parallel lines based on vertices of regular polygons require that the segments have the same number of vertices between the lines on both sides of the polygon (this follows from the *Inscribed Angle Theorem*). This can only occur if ***n*** is even.

How far apart are these lines for even ***n***? The answer is (***n***-2***J***)/2. So, the 12,5-star has parallel lines separated by 1 vertex (1 = (12-2·5)/2), but the 14-5-star parallel lines are separated by 2 vertices on each side of the 14-gon (2 = (14-2·5)/2).

Horizontal Lines. If an ***n***,***J*** star has a horizontal line, then ***n*** is odd. A horizontal line requires the same number of vertices spanned on both sides of the vertical center line. If ***n*** is even, there is a vertex at ***n***/2 so that ***J*** would have to be even. But ***n*** and ***J*** both even means GCD(***n***,***J***) ≥ 2 so that this star cannot be continuously-drawn. By contrast, if ***n*** is odd and ***J*** is coprime to ***n***, then there will be a single horizontal line in the image. If ***J*** is odd, that line is below the center of the polygon (like 11,5 or 11,3 or 11,1) and if ***J*** is even, it is above the center (like 11,4 and 11,2).

Internal Polygons. An early *Polygons Exercise* notes that if ***n*** is odd, the ***n***-gon has a flat bottom and if ***n*** is even, the bottom is pointed because we always start at a pointed top (by construction). The same is not true with internal polygons. All even internal ***n***-gons have pointed bottoms but odd internal ***n***-gons have a flat top if ***J*** is even because the flat line is above the center. Again, compare 11,5 or 11,3 with 11,4 or 11,2 to see this internal 11-gon distinction.