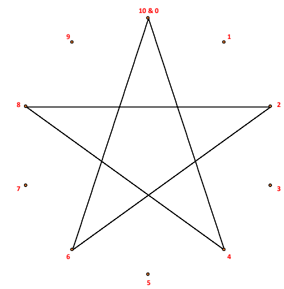
Not all Even Stars are Created Equal

The sharpest angle that can be created from an ***n***-gon depends on whether ***n*** is even or odd. As discussed in the *Sharpest Stars* explainer, the sharpest star using an odd value of ***n*** spans a single vertex of the ***n***-gon. By contrast, the sharpest star based on even ***n*** spans two vertices because even stars have a point at the bottom (at ***n***/2) so the largest number less than ***n***/2 is a whole number (1) smaller rather than 1/2 when ***n*** is odd.



One can always set ***J*** = ***n***/2 – 1 to create the sharpest angle based on an even ***n***-gon but this DOES NOT guarantee that the ***n***,***J***-star thus created will have ***n*** points. Indeed, *half the time, it has half as many points* (as this 10,4-star shows us this star is actually a pentagram).

To see why this is the case and to obtain a general rule for the sharpest ***n***-star, we need to consider two different types of even numbers: those that are divisible by 2 but not 4 and those that are divisible by 4.

n **divisible by 4.** The first star divisible by 4 has 8 points so it is convenient to use ***n*** = 4***k***+4 to represent these values. Half-way around is ***n***/2 = 2***k***+2 and one less is ***J*** = 2***k***+1 which is **ALWAYS an ODD** number.

CLAIM: GCD(4***k***+4, 2***k***+1) = 1 for all ***k*** ≥ 1.

Rationale: All factors of ***n*** = 4***k***+4 are also factors of ***n***/2 = 2***k***+2 (except one less power of 2). Therefore, if ***n*** is divisible by some other number ***f*** (such as ***f*** = 3 to take the smallest example) then so will be 2***k***+2. This means that 2***k***+1 cannot be divisible by ***f*** (the next smaller value divisible by ***f*** is 2***k***+2-***f***, which for ***f*** = 3, would be 2***k***-1).

In this instance, since ***n*** and ***J*** have no common factors, this value of ***J*** will be the sharpest ***n***,***J***-star with ***n*** points.

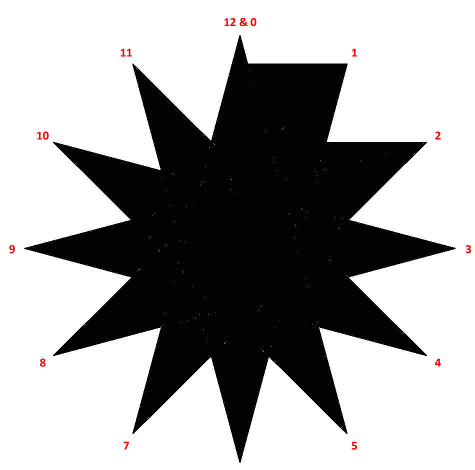
n **divisible by 2 but not 4.** The first star divisible by 2 but not 4 has 10 points (there is no continuously-drawn 6-point star) so it is convenient to use ***n*** = 4***k***+6 to represent these values. Half-way around is ***n***/2 = 2***k***+3 and one less is ***J*** = 2***k***+2 which is **ALWAYS an EVEN** number. This poses a problem because GCD(4***k***+6, 2***k***+2) = 2 so that even though we can create the ***n***,***J***-star, it will have ***n***/2 points (like the 10,4-star shown above).

The sharpest ***n***-star in this instance occurs when ***J*** is 2 less than ***n***/2 or ***J*** = 2***k***+1. Note that this value of ***J*** is necessarily odd, and it has no factors in common with ***n*** using the above *Rationale*. Note that this angle spans 4 vertices rather than 2 when ***n*** is divisible by 4, or 1 when ***n*** is odd.

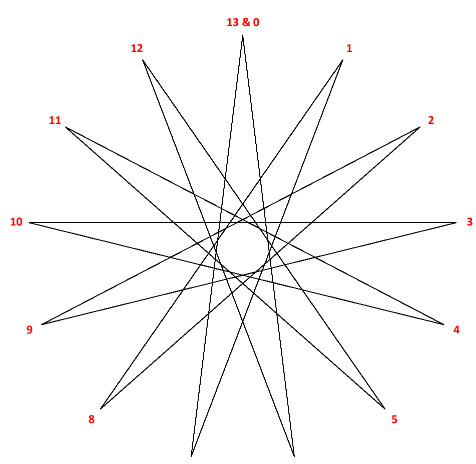
Angle measures. To recap and put this in terms of angles (and organized according to the three images below):

The sharpest angle possible in an ***n***-point star = (the number of spanned vertices)·180°/***n***, or:

360°/***n*** if ***n*** is divisible by 4.
n = 12, (360/12 = 30°)



180°/***n*** if ***n*** is odd.
n = 13, (180/13 = 13.85°)



720°/***n*** if ***n*** is divisible by 2 but not 4.
n = 14, (720/14 = 51.43°).