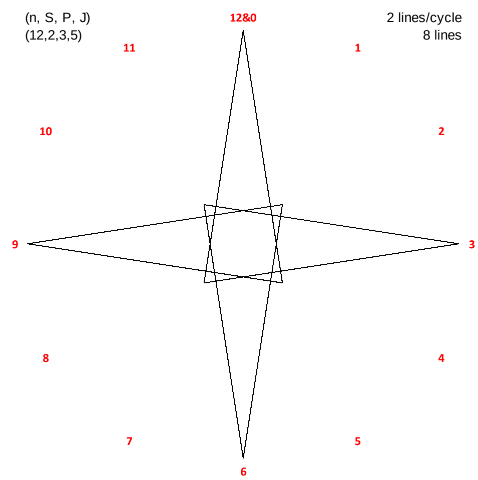
Rotational Symmetry versus Lines of Symmetry

Images in the String Art model based on a uniform jump pattern (of ***J*** vertices of the ***n***-gon per vertex frame line added) exhibit vertical symmetry. This means that one could imagine taking an image and carefully cutting it in half along the vertical line through the top and the center, and then place this half against a mirror at right angle to the mirror. The half image plus its reflective half together create the original image (once you adjust your viewing location to a 45°).

We also know that the first cycle tells us the essential character of the image since the completed image is simply made up of rotated versions of this cycle. As noted there, there are ***C*** = ***S***/GCD(***S***,***P***) lines per cycle and ***M*** = ***Lines***/***C*** cycles in the final image (***Lines***, ***C*** and ***M*** are noted in cells M3, M9 and M11 in the Excel file). The number of rotations is the same as the number of vertices used in the final image (which is also the same as the number of cycles in the image, ***M***).

The Number of Cycles Determines the Number of Lines of Symmetry. We know that the initial image has vertical symmetry. But when the number of cycles in the image ***M*** > 1, then there is a line of symmetry through each used vertex and the center. The reason we can say this is because we know that by rotating the image by 360°/***M*** we have the same image and if we rotate another 360°/***M*** we have the same image and so on. The number of vertices *between* used vertices is ***n***/***M*** (at right, 12/4 = 3, every 3rd vertex is used). It is worth considering even versus odd ***M*** values separately.

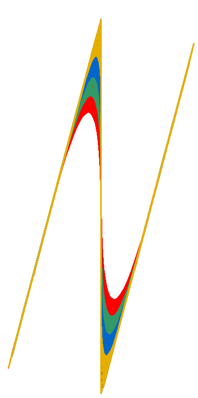
Odd M. To take a very simple example, set ***S*** = ***P*** = 1 and consider a 10,4-star versus a 5,2-star. Both are created in 5 one-line cycles so ***M*** = 5 and the same image occurs each time the image is rotated 360°/5 = 72°. There are 5 lines of symmetry because one can draw distinct lines through each used vertex and the center, whether or not it touches a vertex on the other side of the circle. For example, the 5 lines of symmetry in the 5,2-star go through the vertices 0 – 4 and the midpoint between the two opposing vertices: 0-2.5, 1-3.5, 2-4.5, 3-0.5, and 4-1.5. Of course, the 10,4-star has even used vertices with lines of symmetry just double each of the lines noted in the prior sentence: 0-5, 2-7, 4-9, 6-1, and 8-3 since VCF = 2. Either way, it is clear that there are ***M*** distinct lines of symmetry when ***M*** is odd.



Even M. When ***M*** is even, you have to be a bit more careful. First, if ***M*** is even, then ***n*** must be even as well because ***M*** is a proper divisor of ***n***. If ***n*** is even, and ***M*** is even, then ***n***/2, half-way around the circle opposite ***n*&0** is also a used vertex. But its line of symmetry is THE SAME AS the vertical line of symmetry. And if ***M*** = 4 then the line of symmetry through ***n***/4 and 3***n***/4 also coincide. The image to the right shows lines of symmetry from **0**-**6** and **3**-**9**.

More generally, let ***M*** = 2***k***. Then 0 and ***k***, 1 and ***k***+1, …, ***k***-1 and 2***k***-1 are all pairs of lines sharing the same line of symmetry. ***k*** such lines exist (from 0 to ***k***-1) so you might think that there are half as many lines of symmetry. But we this ignores the other half of the lines of symmetry when ***M*** is even. The lines from ½ to ***k***+½, 3/2 to ***k***+3/2, …, ***k***-½ to 2***k***-½ are also lines of symmetry. There are ***k*** such lines ***at-the-halves***. To use the image at right, the lines from **1.5** to **7.5** and **4.5** to **10.5** are lines of symmetry. In total, there are ***M*** lines of symmetry whether ***M*** is even or odd.

When are lines perpendicular to lines of symmetry symmetric? This is true when ***M*** is even, but not when ***M*** is odd.



A hint of things to come: Multiple Jump Sets. The image to the left does NOT have vertical symmetry, nor is there symmetry through any other line. But it does have rotational symmetry. If you flip the image over, you get the same image. Single jump pattern string art images exhibit both rotational and line symmetry, but multiple jump set models may well exhibit rotational symmetry without linear symmetry. We do not want to focus too much on multiple jump set models here, but a simple example will show the difference. The 4-color image was created from ***n*** = 12 so we can do some clock arithmetic (so we use **12** here rather than **12&0** or **0**). Consider the following 3-jump-set pattern: 7, 5, 6. The 6-line VF of the final image is **12**-**7**-**12**-**6**-**1**-**6**-**12**. Four of the 12-gon's vertices are used in creating this image but note that vertices **7** and **1** are only used once but vertex **6** is used twice. The resulting image has 180° rotational symmetry but there is NO line of symmetry.