Why the *Inscribed Angle Theorem* works

There are really two surprising points about the *Inscribed Angle Theorem*.

> **A.** The inscribed angle is exactly half the size of the central angle.
> **B.** It doesn't depend on which point you choose for the common vertex of the angle as long as it is on the circle.

You can see the rationale for both points by considering the possible locations for the common vertex. There are three possibilities. **I.** One line is a diameter of the circle. **II.** The two lines "surround" the center of the circle. **III.** The angle created by the two lines excludes the center of the circle. The three possibilities are shown in the images to the right.

We start with three facts from geometry.

a. An *isosceles* triangle has two equal sides, traditionally called *legs*. The third side is called the *base*. The angle formed by the two legs is called the *vertex angle* and the other two are *base angles*. The base angles are equal.
b. The sum of angles in a triangle is 180°.
c. The sum of adjacent angles constructed when two lines intersect is 180°. These angle pairs are often called *supplementary angles*.

*A point about notation***.** In all three images, the angle under consideration is $\angle$***vsw***. The lines that create the angle, ***sv*** and ***sw***, are in black, and all other identifying letters and lines are in blue. The letter ***r*** stands for radius.

I. In Figure I, angle ***a*** has one line (***sw***) which is a diameter of the circle. The midpoint of this line is ***c***, the center of the circle. Two sides of Δ***scv*** are ***r*** so the triangle is isosceles with apex angle ***b*** and base angles of ***a***. The sum of triangle angles is ***b*** + 2***a*** = 180°.

Line ***vc*** creates two supplementary angles, ***b*** and $\angle$***vcw*** = 180° - ***b*** which is a central angle. We know that ***b*** + 2***a*** = 180° so $\angle$***vcw*** = 2***a***. The inscribed angle spanning ***vw*** is half the size of the central angle spanning these same points.

II. By moving ***s*** in Figure **I.** counterclockwise by a bit one could obtain an angle in which the two lines surround the center, but the angles would be smaller, and labelling would be harder. As a result, $\angle$***vsw*** is larger in Figure **II** than in **I**. The diameter through ***s*** intersects the circle at ***y***. This creates two angles, ***a*** and ***d*** that sum to $\angle$***vsw***.

Applying the argument made in **I.** twice we obtain the central angle spanning ***vw*** is *2**a** + 2**d*** = 2·(***a*** + ***d***) or twice the inscribed angle of ***a*** + ***d***.

III. The point ***s*** was chosen so that $\angle$***vsw*** excludes the center in Figure **III**. Construct a diameter through ***s*** that intersects the circle at ***y***. Construct a radius at ***v***. The newly constructed isosceles triangle ***scv*** has base angle of ***b*** = ***csv*** and central angle of $\angle$***ycv*** = 2***b*** using the argument presented in **I.** above.

Construct a radius at ***w***. The newly constructed isosceles triangle ***scw*** has base angle ***a*** + ***b*** = $\angle$***csw*** and central angle 2·(***a*** + ***b***) = $\angle$***ycw*** using similar reasoning.

Since the central angle 2·(***a*** + ***b***) = 2***a*** + 2***b*** = $\angle$***ycw*** = $\angle$***ycv*** + $\angle$***vcw*** and because $\angle$***ycv*** = 2***b***, we see that central angle $\angle$***vcw*** = 2***a*** by subtraction.

Wherever the vertex is placed, the inscribed angle is half the central angle.