Are Almost Polygon and Star Sides Parallel? (Take 2)

Adjust n and J so that GCD(n, J) = 1. Let (a, b) and (c, d) be the coordinates of two parent polygonal vertices that are J jumps apart. Assume the line in question is the k^{th} line of the spiral and further assume that $k+n < r$ so that there is a next almost polygon line inside the one in question.

To make this explicit, consider $k = 4$, the 4^{th} line in the almost pentagon and pentagram shown below. For the pentagon the starting point is $3/50^{th}$ inside vertex 3 and ending point is $4/50^{th}$ inside vertex 4 and for the pentagram it is $3/50^{th}$ inside vertex 1 to $4/50^{th}$ inside vertex 3.

More generally, the k^{th} line starting point is $((r\text{-}k+1)a/r, (r\text{-}k+1)b/r)$ and the k^{th} line ending point is $((r\text{-}k)c/r, (r\text{-}k)d/r)$. The slope of this line is: $m_k = [(r\text{-}k)d/r - (r\text{-}k+1)b/r]/[(r\text{-}k)c/r - (r\text{-}k+1)a/r] = [(r\text{-}k)d - (r\text{-}k+1)b]/[(r\text{-}k)c - (r\text{-}k+1)a]$.

The line just inside this line is the $k+n^{th}$ (or the 9^{th} line in the images below given our $k = 4$ and $n = 5$ example). The slope of this line is: $m_{k+n} = [(r\text{-}k\text{-}n)d - (r\text{-}k\text{-}n+1)b]/[(r\text{-}k\text{-}n)c - (r\text{-}k\text{-}n+1)a]$.

These two lines are parallel if the difference in slopes is zero. This difference requires creating a common denominator. The numerator of $m_k - m_{k+n}$ must be zero for the difference to be zero. This numerator is:

$$[(r\text{-}k)d - (r\text{-}k+1)b]\cdot[(r\text{-}k\text{-}n)c - (r\text{-}k\text{-}n+1)a] - [(r\text{-}k\text{-}n)d - (r\text{-}k\text{-}n+1)b]\cdot[(r\text{-}k)c - (r\text{-}k+1)a]$$

$$= [(r\text{-}k)(r\text{-}k\text{-}n)cd - (r\text{-}k)(r\text{-}k\text{-}n+1)ad - (r\text{-}k+1)(r\text{-}k\text{-}n)bc + (r\text{-}k+1)(r\text{-}k\text{-}n+1)ab] -$$
$$[(r\text{-}k)(r\text{-}k\text{-}n)cd - (r\text{-}k\text{-}n)(r\text{-}k+1)ad - (r\text{-}k)(r\text{-}k\text{-}n+1)bc + (r\text{-}k+1)(r\text{-}k\text{-}n+1)ab]$$

The highlighted terms cancel, and the middle terms simplify to $n(bc - ad)$, which is only zero if the two parent polygon vertices are either the same or opposite one another on the unit circle. Since the points are distinct, the first is not the case, and the second occurs only if $J = n/2$ (which is precluded since n and J are assumed to be coprime).

[**MA.** This assertion is based on two facts: the two points are on the unit circle and hence can be thought of in terms of trigonometric coordinates (a, b) = (cosα, sinα) and (c, d) = (cosθ, sinθ). Given this, by the angle subtraction formula, sin(α-θ) = sinα·cosθ – cosα·sinθ, the term $bc - ad$ is seen to equal zero only if θ = α or θ = α+π.]