A Fractional Analysis of Full Density Images

Using a screening method similar to the Sieve of Eratosthenes, E21.1.2, we see a visual method for obtaining the number of full density images in an image sequence. That analysis focused on single jump models but jump sets are also amenable to the strategy shown there. Here we examine an algebraic method of finding the same number.

Background. This method keys off the number of possible ***P*** values which are coprime to ***n***, ***k***, and ***S*** where ***k*** is the number of jumps in the jump set. The product of ***n*** and ***k*** is the number of vertex endpoints used in the vertex frame, VF, if each is used ***k*** times. This occurs when VCF = 1. In the jump set case, VCF = GCD(***m***, ***n***), where $\boldsymbol{m} = \boldsymbol{J_1}+\ldots+\boldsymbol{J_k}$, as noted in E16.1. The number of vertices used (sometimes multiple times), $\boldsymbol{V_{used}}$, is $\boldsymbol{V_{used}} = \boldsymbol{k}\cdot\boldsymbol{n}/\text{VCF}$, and the number of subdivisions, ***T***, is $\boldsymbol{T} = \boldsymbol{S}\cdot\boldsymbol{V_{used}}$. Full density images require all ***T*** subdivisions are used in the final image or SCF = GCD($\boldsymbol{S}\cdot\boldsymbol{V_{used}}$, ***P***) = 1 according to E5.4.2. Since the total number of subdivisions varies as VCF varies, it is helpful to think in terms of the fraction of subdivisions that are coprime to ***T***. The question is: *How many full density images are in an image sequence?*

The prime factors of *T*. Suppose there are ***b*** distinct prime factors of ***T***. Label these factors $\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_b}$. As noted elsewhere, It may be the case that there is only one prime factor in ***T***.

Single Prime Example 1. The (8,32,***P***,***J***(3,2)) sequence has 512 lines. Note that ***n***, ***S***, and ***k*** are all powers of $\boldsymbol{a_1}$ = 2. The VF has an 8,2-star and an 8,3-star superimposed on one another. By manually adjusting ***P***, one quickly notices that every even ***P*** has SCF > 1, but all odd ***P*** have SCF = 1. In all, there are 256 full density images (but only half are unique since there is symmetry about the porcupine value of ***P*** = 255 (255 and 257 = 512-255 are the same image). Change jumps to (8,32,***P***,***J***(5,1)) and one obtains a square with eyelets VF. Each full density image has 256 lines (since VCF = 2). There are 128 full density images (but only half are unique since there is symmetry about the porcupine value of ***P*** = 127 (127 and 129 are the same image). In both instances, 1/2 = $(\boldsymbol{a_1}\text{-}1)/\boldsymbol{a_1}$ of the ***P*** values are full density.

Single Prime Example 2. The (9,27,***P***,***J***(3,4,1)) sequence has 729 lines. Note that ***n***, ***S***, and ***k*** are all powers of $\boldsymbol{a_1}$ = 3. The VF has a 9,4-star, a 9,3-star, and a 9-gon superimposed on one another. By manually adjusting ***P***, one quickly notices that every third ***P*** has SCF > 1, but all other ***P*** have SCF = 1. In all, there are 486 full density images (but only half are unique since there is symmetry about the porcupine value of ***P*** = 364 (364 and 365 are the same image). Change jumps to (9,27,***P***,***J***(3,8,1)) and one obtains a triangle with tails VF (E17.3). Each full density image has 243 lines (since VCF = 3). There are 182 full density images (but only half are unique since there is symmetry about the porcupine value of ***P*** = 121 (121 and 122 are the same image). In both instances, 2/3 = $(\boldsymbol{a_1}\text{-}1)/\boldsymbol{a_1}$ of the ***P*** values are full density.

A Two Prime Example. The (5,27,***P***,***J***(2,1,5)) sequence has 405 lines. The VF includes a pentagram, a pentagon, and a zero-jump at each vertex (as discussed in E17.4.2). Here $\boldsymbol{a_1}$ = 3 and $\boldsymbol{a_2}$ = 5. ***P*** values which have remainder 1, 2, 4, 7, 8, 11, 13, and 14 upon division by 15 are coprime to **any** ***T*** which is composed of only powers of 3 and 5.

Therefore, 8 out of 15 ***P*** are coprime to ***T*** or $8/15 = 2/3\cdot4/5 = (\boldsymbol{a_1}\text{-}1)/\boldsymbol{a_1}\cdot(\boldsymbol{a_2}\text{-}1)/\boldsymbol{a_2}$. Given ***T*** = 405, there are 216 = 405·8/15 full density images (but only half are unique since there is symmetry about the porcupine value of ***P*** = 202 (202 and 203 are the same image). Change jumps to (5,27,***P***,***J***(2,1,2)) and an isosceles triangle VF results with 81 lines. VCF = 5 creates a situation where three vertices are used. The resulting ***T*** =81 is a power of 3 and there are 54 = 2/3·81 full density images which are symmetric about the porcupine value of ***P*** = 40.

A General Rule. Given ***b*** distinct prime factors of ***T***, labeled $\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_b}$ the number of full density images in an image sequence is:

$$\text{Full density images} = \boldsymbol{T}\cdot(\boldsymbol{a_1}-1)/\boldsymbol{a_1}\cdot(\boldsymbol{a_2}-1)/\boldsymbol{a_2}\cdot \ldots \cdot(\boldsymbol{a_b}-1)/\boldsymbol{a_b}.$$

Half of these images are unique because the images are symmetric about the porcupine value of ***P***.

Smooth Image Sequences. The fraction of full density images will be larger, and the image sequence will appear to morph more smoothly, if ***T*** has a smaller number of prime factors or if the primes are larger. Consider three image sequences: (15,14,***P***,4), (15,13,***P***,4), and (13,13,***P***,4). The first has 48 (22.9% of all ***P***) 210-line images, the second has 96 (49.2% of all ***P***) 195-line images, and the third has 156 (92.3% of all ***P***) 169-line images. Even though the third involves the fewest lines, it has a much smoother feel as a result. Finally, note how smoothly (11,19,***P***,4) or (5,31,***P***,2) adjust due to the larger prime values of ***S***. One has 180 (86.1%) 209-line images, the other has 160 (78%) 205-line images.