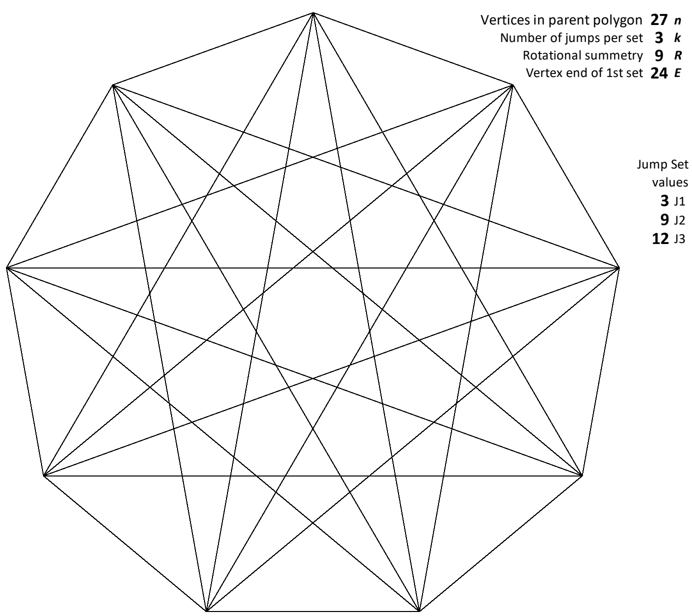
On the Relative Size of Regular Polygons and Stars in Jump Set Vertex Frames

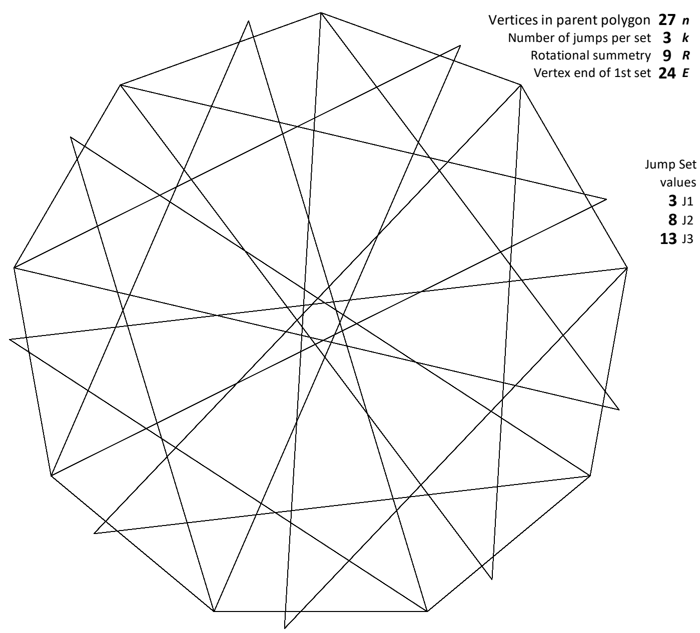
The rotational symmetry of the VF defines what kinds of regular polygons and polygrams are visible in the VF. For simplicity we are going to call both stars since a polygon occurs if ***J*** = 1 and a polygram occurs if ***J*** > 1 so calling them all stars provides a convenient shorthand. Here we are only interested in those stars that are centrally located.

Three types of stars are possible, inscribed, interior, and truncated. These two images show this distinction, using **PwP** file 12b. The only difference is one less vertex for J2 and more for J3 on the right (3,8,13) than on the left (3,9,12) based on ***n*** = 27. For both, the jump set sum is 24 and VCF = GCD(27,24) = 3 so the image has 9 = ***n***/VCF degrees of rotational symmetry. In the images below, the degree of rotational symmetry is noted by ***R***, ***R*** = ***n***/VCF.

NOTE: Here we have a bit ambiguity in that we typically talk about ***n***,***J***-stars rather than ***R***,***k***-stars. We focus here on stars that are a factor of ***n***, rather than ***n***. In the context of these two images, our interest is in 9,***k***-stars even though ***n*** = 27.

Both show three stars, 9,1, 9,3 and 9,4 (which are noted without the *-star* for ease of discussion). In the left image, all three stars are inscribed; in the right, the 9,4-star is interior, the 9,3-star is truncated, and the 9,1-star (nonagon) is inscribed. As discussed in ESA Sections 2.6.1 and 2.6.2, any ***n***,***J*** star also includes interior ***n***,***J***-1, ***n***,***J***-2, … ***n***,1 interior stars.





Interior and inscribed stars are quite easy to see, but truncated stars require a bit more focus.

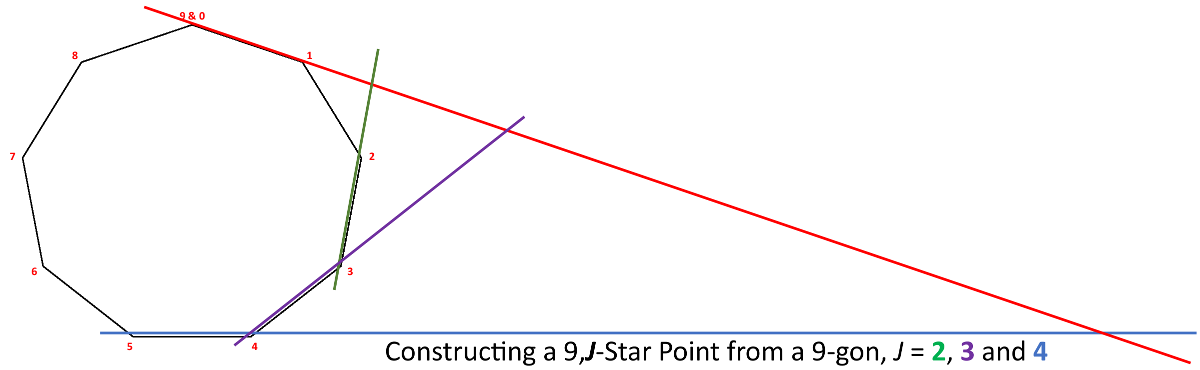
Truncated Stars. The three equilateral triangles making up the inscribed 9,3-star on the left are easy to see. Each is tilted 360/9 = 40° to one another. The highest peak for each triangle is at vertices 24, 0 (= 27), and 3 of the 27-gon. By contrast, the truncated 9,3-star on the right has (truncated, or missing) peak vertices between the 27-gon vertices 23-24, 26-0, and 2-3. Consider only the 26-0 missing vertex. That peak is at the intersection of the extension of the line from vertex 18 to 26 and from vertex 8 to 0. Even though these lines do not touch, the implied exterior angle represented by these lines is 60° (ESA Section 22.3).

A *Numerical Rule*. Let j_i = minimum(J_i,n-J_i) where J_i is the i^{th} jump in the jump set. If J_i = n/2 the jump is a diameter of the circle, otherwise, $j_i < n/2$. The type of ***R***-gon produced by an individual jump, J_i, is determined by the relative size of VCF and j_i. The same rule applies to ***R***,***k***-stars. The general rule is stated both for polygons and stars as:

> If j_i (<, =, >) VCF, a (truncated, inscribed, interior) ***R***-gon results.
> If j_i (<, =, >) ***k***·VCF, a (truncated, inscribed, interior) ***R***,***k***-star results.

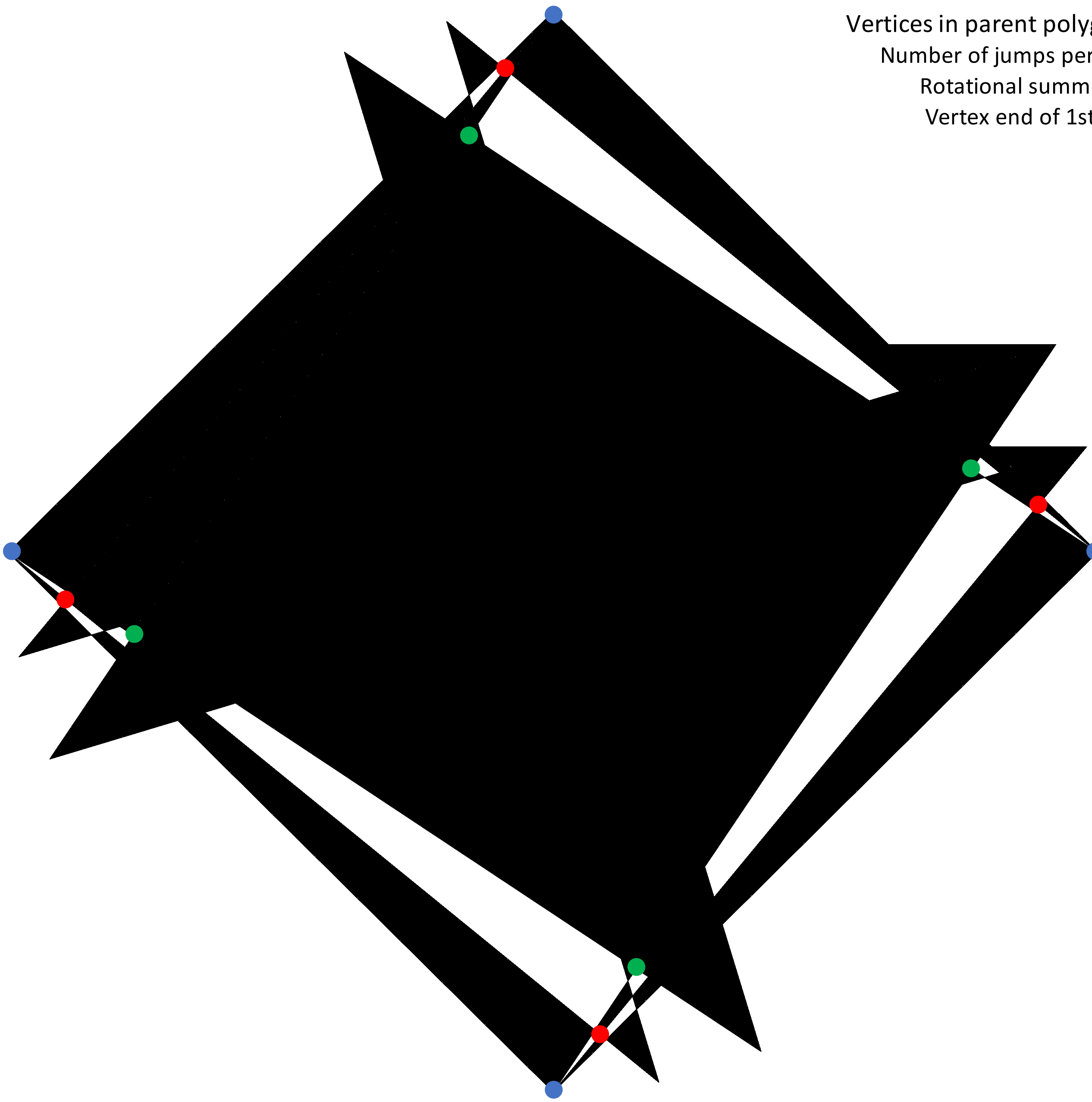
Notice that the above images are consistent with this rule. Given a 27-gon and VCF = 3, the left image with jump set (3,9,12) has inscribed 9,1, 9,3 and 9,4 stars and the right with jump set (3,8,13) has inscribed 9,1, truncated 9,3, and interior 9,4-stars.

Constructing Truncated Stars from a Regular Polygon. This shows how to create stars from polygons. The **red**, **green** intersection is a vertex of a 9,2-star, the **red**, **purple** intersection is a vertex of a 9,3-star and the **red**, **blue** intersection is a vertex of a 9,4 star. The other 9 vertices are obtained in similar fashion simply by extending lines outward from each side. Think of this as inverting the material discussed in E2.6.1 and E2.6.2 since it looks outward rather than inward.



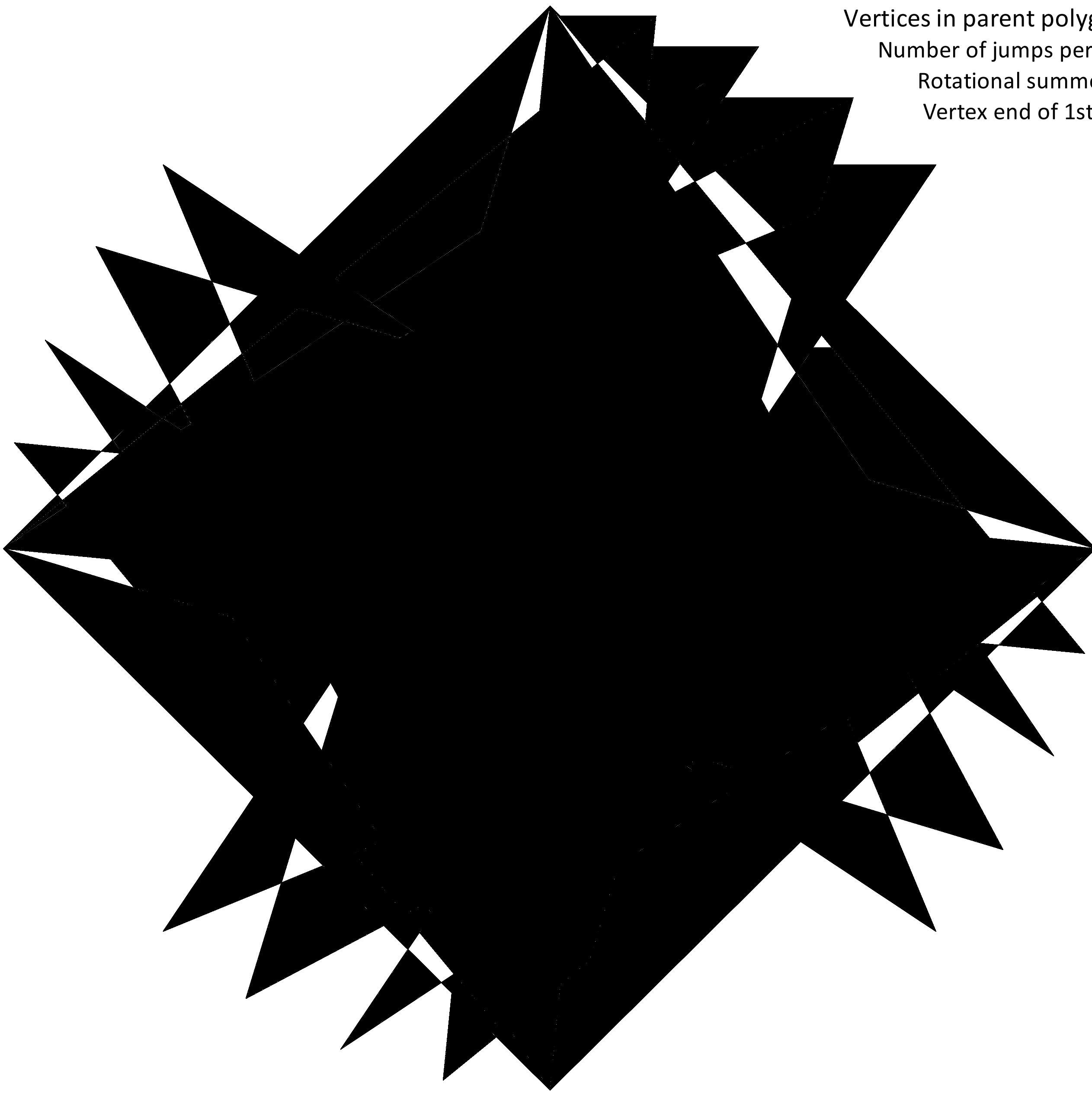
Constructing a 9,***J***-Star Point from a 9-gon, *J* = **2**, **3** and **4**

Squares of Various Sizes. The next 3 images examine centrally located squares; each having the same inscribed square.



The above image shows 4 squares, the smallest, and the three largest (that are not truncated) given ***n*** = 32. The smallest is based on a jump of 15 = ***n***/2-1, the largest is inscribed and is based on a jump of 8 = ***n***/4 (**blue dots** at vertices 0, 8, 16 and 24). The second largest (**red dots**) is based on a jump of 9 and the third largest (**green dots**) is based on a jump of 10.

Vertices in parent polygon **32** ***n***
Number of jumps per set **8** ***k***
Rotational summetry **4** ***R***
Vertex end of 1st set **24** ***E***

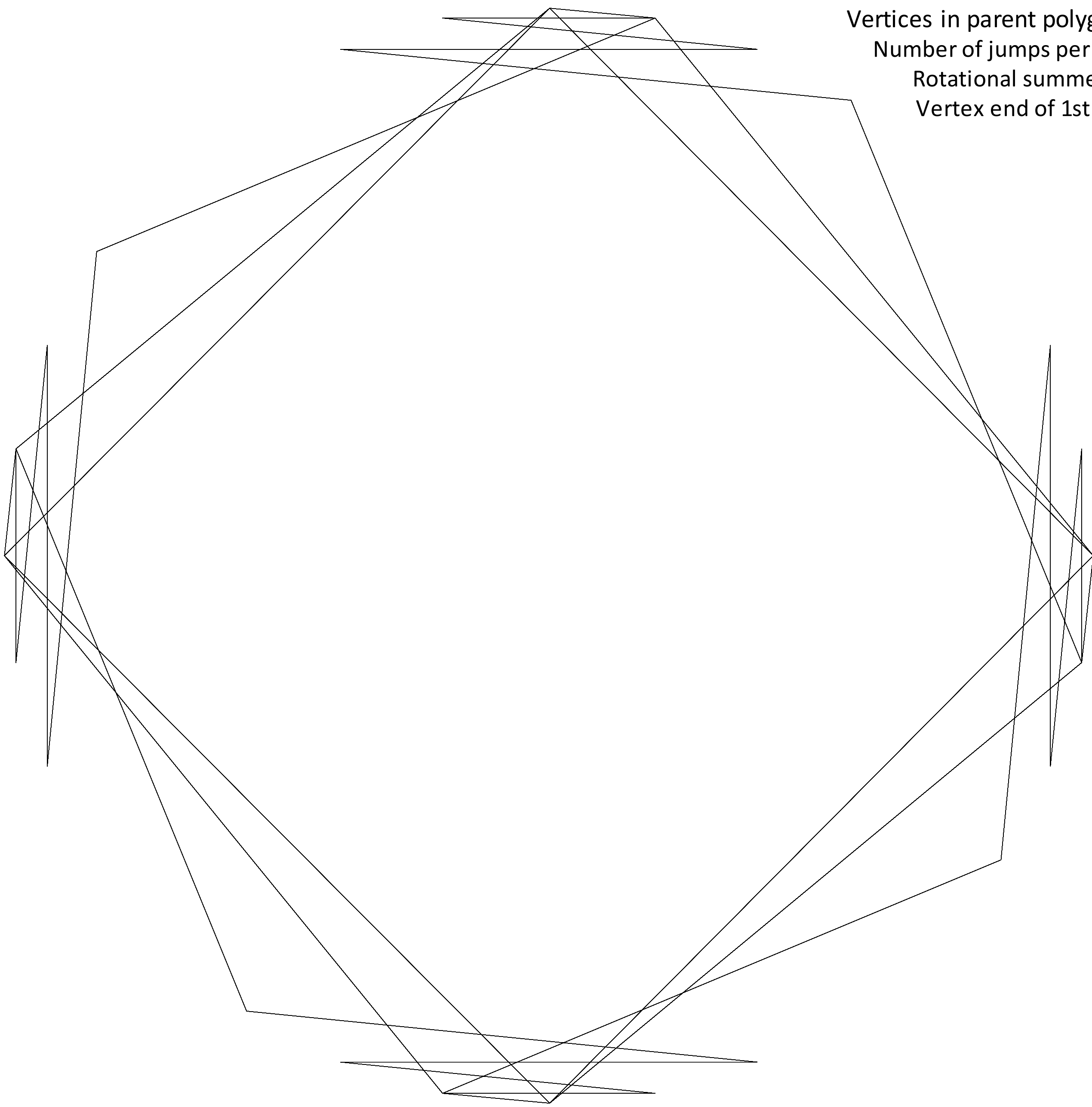


Jump Set values
9 J1
15 J2
24 J3
10 J4
18 J5
20 J6
11 J7
13 J8

The second squares image shows 7 internal and one inscribed central squares based on ***n*** = 32. The largest is inscribed, J3 = 24, which is a jump of 8 vertices since jump size is determined by minimum(***J***,***n***-***J***); the rest are interior because each jump is in the range ***n***/4 = 8 < ***J*** < 24 = 3/4***n***. Each jump size determines a square of a different size. As with the previous image, the largest square is ***J*** = 8 and the smallest is ***J*** = ***n***/2-1 = 15.

This jump set was chosen because 7 of the 8 squares have a side extension that passes through the vertices 0, 8, 16, and 24. Consider the top RIGHT side starting at vertex 0. The largest is the inscribed square with top right side between vertex 0 and 8. The second largest top right side is from 0 to 9, third from 0 to 10, fourth from 0 to 11, fifth from 0 to 12. The sixth largest has top LEFT side from 0 to 19 which is a jump of 13 since jump size is minimum(19, 32-19). The eighth smallest has top LEFT side from 0 to 17 which is a jump of 15 = minimum(17, 32-17). The only square that does not have a side attached to vertex 0 is the seventh smallest. The top LEFT from 2 to 20 is a jump of 14 = minimum(18, 32-18).

The final squares image shows eight larger squares, all but one of which are truncated. The first 5 jumps set up a clear zig-zag pattern from vertices 0-1-31-2-30-3. This zig-zag pattern produces two sets of multiple parallel nested truncated squares in two orientations. One orientation has horizontal and vertical sides. The horizontal and vertical pair have tops that extend from the horizontal lines 1-31 and 2-30. These truncated squares are based on jump size 2 and 4, respectively. The other three parallel nested truncated squares have top sides that are at the shallowest downward sloping steepness possible for the top side given ***n*** = 32, 180/32 = 5.625°. These tops are created from extensions of the parallel lines 0-1, 31-2, and 30-3 and are based on jump sizes 1, 3, and 5, respectively. The next smaller truncated square is jump size 6 and has top RIGHT side from vertices 3 to 9. The second smallest is jump size 7 based on the line extending from 9 to 16 on bottom RIGHT. The smallest square in the image is the inscribed square of jump size 8 connecting vertices 0, 8, 16 and 24. It is the only square that is complete, and it is based on the 8th jump in the first set this jump is from 16 to 24.



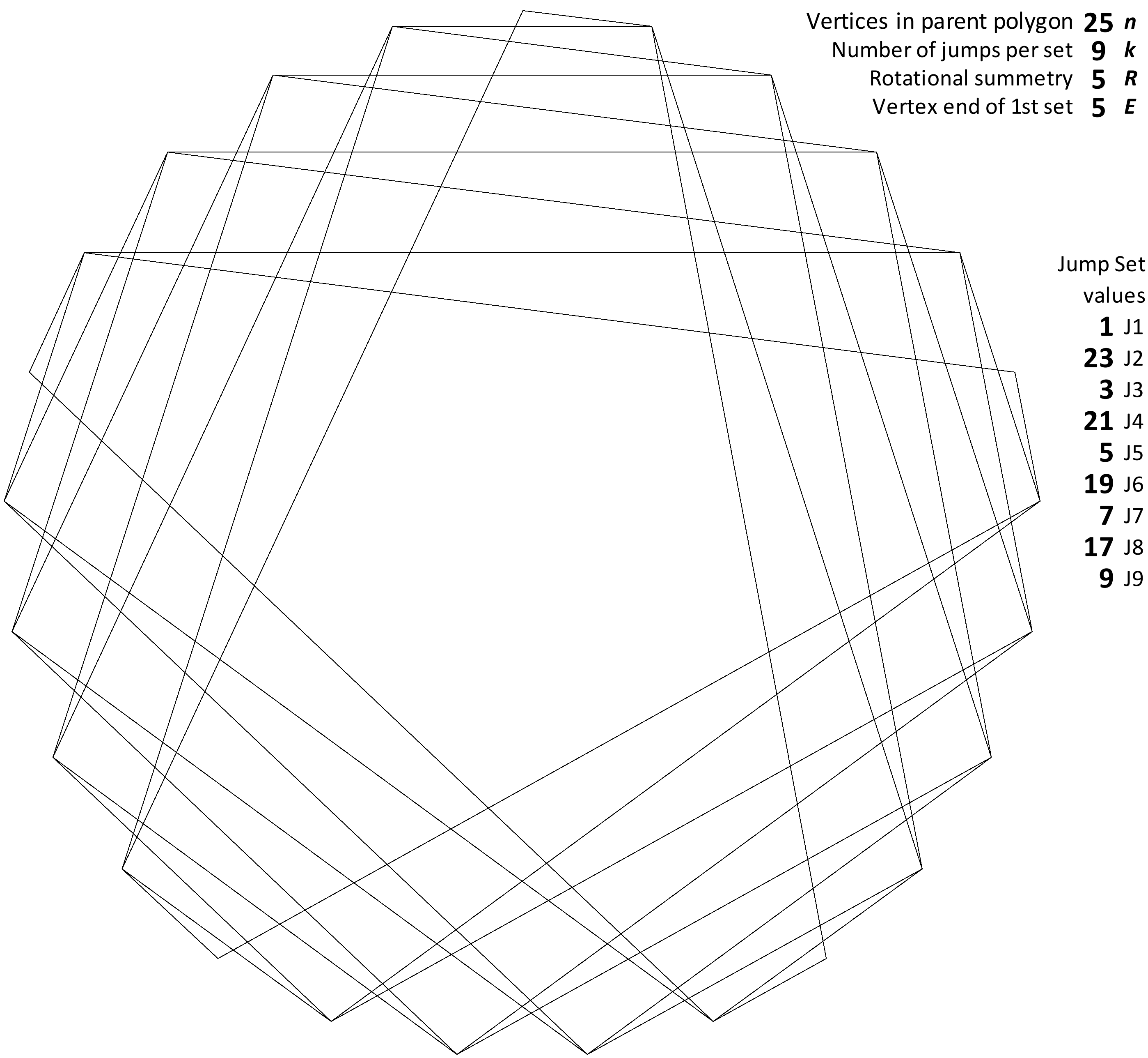
Vertices in parent polygon **32** ***n***
Number of jumps per set **8** ***k***
Rotational summetry **4** ***R***
Vertex end of 1st set **24** ***E***

Jump Set values
1 J1
30 J2
3 J3
28 J4
5 J5
6 J6
7 J7
8 J8

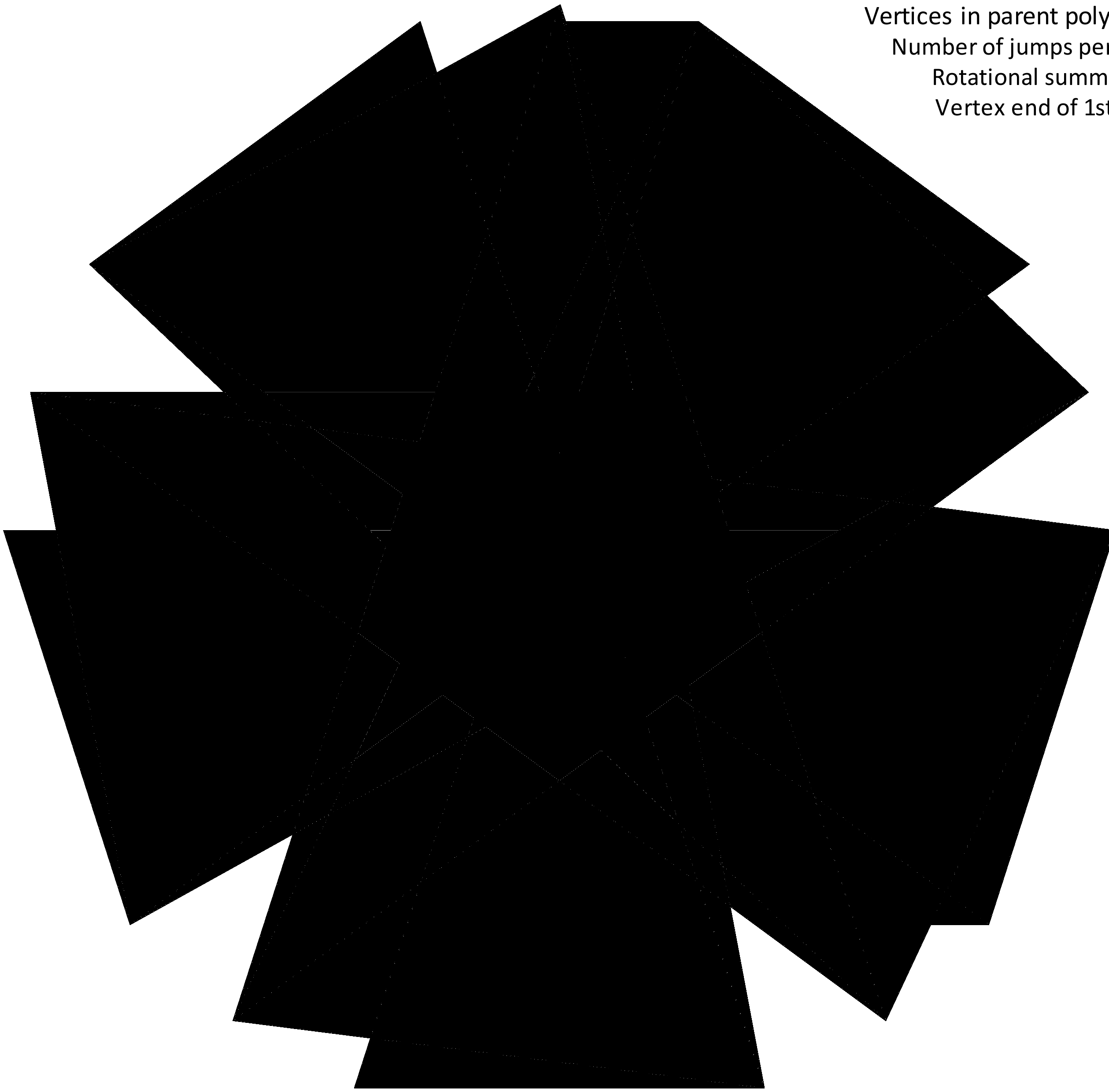
Some of the image sequence images are stunning. This crossed ovals ***P* = 305** is from the second. This folded ribbon ***P* = 287** is from the third. Compare these 1024-line images to the 64-line ***P*** = 304 image and the 32-line ***P*** = 288 image.

Combined, the last two squares images provide examples of all possible sized centrally drawn squares given ***n*** = 32. Fifteen are possible since 15 is the largest number less than ***n***/2. The 7 with jumps below 8 are truncated and larger, the 7 with jumps larger than 8 are interior and smaller. And when ***J*** = 8 = ***n***/4, the square is inscribed and uses four equally spaced vertices of the parent 32-gon, albeit not necessarily at 0, 8, 16, 24 as shown in the above three VF images.

Pentagons and Pentagrams. Jumps from 1 to 9 are shown in this ***n*** = 25 image using the same zig-zag pattern implemented in the truncated squares above. The resulting VF shouts out for different sized truncated 5,2-stars. As with the squares, the zig-zag pattern creates two orientations. The easiest way to consider both is to go back to the standard 5,2-star (based on ***n*** = 5) obtained by connecting vertices 0-2-4-1-3-0 with the third line being horizontal from 4 to 1. There are four horizontal lines in the image below associated with jumps of 2, 4, 6, and 8 (connecting vertices 1-24, 2-23, 3-22 and 4-21). There are five with slightly downward tilting cross-pieces (180/25 = 7.2°) associated with jumps of 1, 3, 5, 7 and 9. Those with jumps less than 5 are also considered truncated pentagons, the 5 jump is an inscribed pentagon (connecting vertices 23, 3, 8, 13, 18 and 23), and those with jumps larger than 5 are interior pentagons. If you add a 10th jump of 10 the resulting jump set remains divisible by 5 since it ends at vertex 15, there are 50 lines in the VF including an inscribed pentagram with peak at vertex 0. Of course, the image sequence produces some interesting images. Note how the zig-zags create five squiggles like at ***P* = 77** or the curved ribbons when ***P* = 461** in the 10 jump version.



The second pentagrams version has 10, 11, 12 jumps balanced by 3 and 4 so that the peak is at 0 for the inscribed pentagram (J1 = 10) and the smallest interior pentagram (J4 = 12). As a result, the cross-piece of the pentagram is a horizontal line in both instances (from vertices 20-5 and from 19-6). The larger interior star (J2 = 11) has a peak between 0 and 1 with cross-piece tilted 180/25 = 7.2° from horizontal (from vertices 20-6). Note that the internal pentagons are easy to see as the middle one is tilted from the other two which have horizontal tops. These tops are parallel to the 3 and 4 jumps at the bottom of the image. The 3 jump bottom is from vertex 11 to 14 and the four jump version is from 11 to 15. Both are parts of truncated pentagons and pentagrams of jump size 3 and 4.



Vertices in parent polygon **25** ***n***
Number of jumps per set **5** ***k***
Rotational summetry **5** ***R***
Vertex end of 1st set **15** ***E***

Jump Set values
10 J1
11 J2
3 J3
12 J4
4 J5

Just like the last two squares VF images, these two pentagonal VF images show all possible sizes of pentagrams and pentagons given ***n*** = 25. Given this ***n***, there are 12 possible sizes of pentagrams and pentagons since 12 is the largest ***J*** less than ***n***/2. These pentagons and pentagrams follow the *Numerical Rule* described above. When the jump size is less than 5, the pentagons are truncated, when greater than 5, they are interior and when jump size is 5, the pentagon is inscribed. And when the jump size is less than 10, the pentagrams are truncated, when greater than 10 the pentagrams are interior, and when jump size is 10, the pentagram is inscribed.