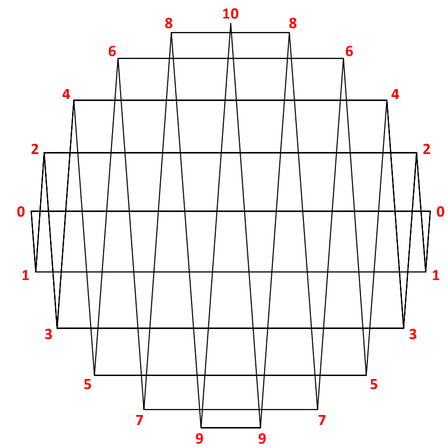
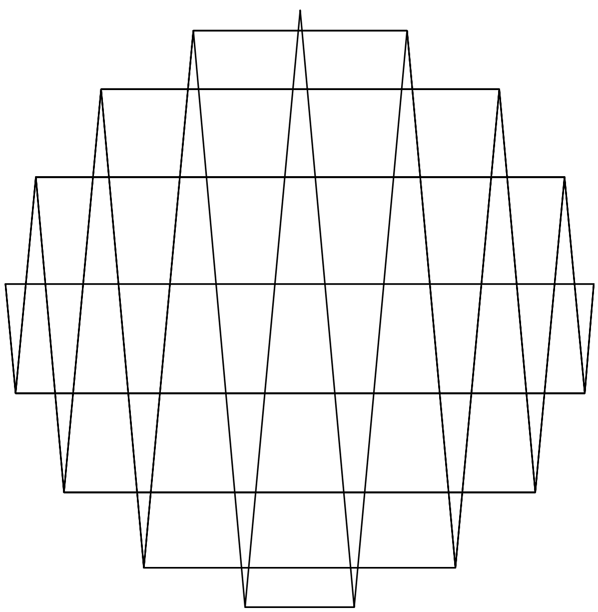
A One Sentence Answer: Look at Apex Counts Zig-Zag from Side-to-Side

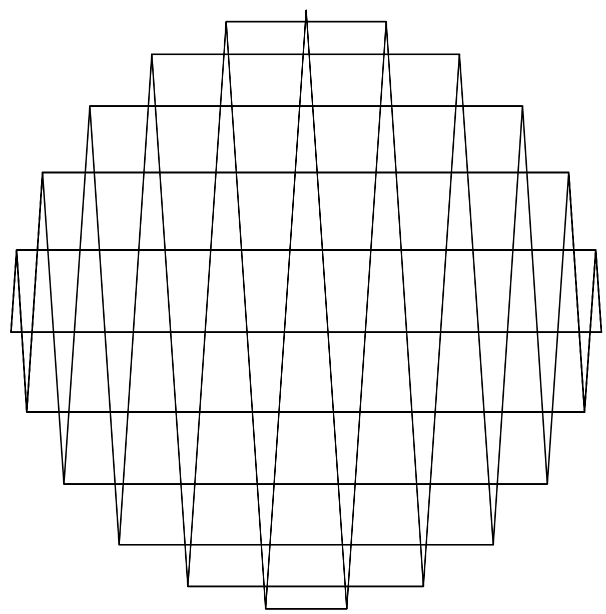
Armed with *The Hill Formula*: $1+2+\ldots+ \boldsymbol{n}-1 + \boldsymbol{n} + \boldsymbol{n}-1 + \ldots+2+1 = \boldsymbol{n}^2$, we see that the apex counts from sharpest isosceles triangles on odd polygonal vertices follows exactly this pattern IF WE COUNT ZIG-ZAG, ALONG THE DIAGONALS. To use the image to the right, start at **0** (on either side), then proceed to **1**, **2**, … until you get to **10** and then go down the other side until you get to **1**, **0**. The image has 100 triangles since **10** is the top of the hill.

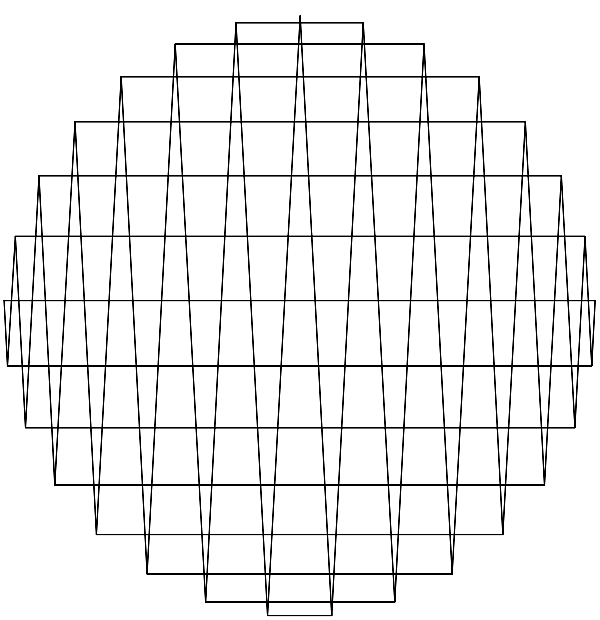


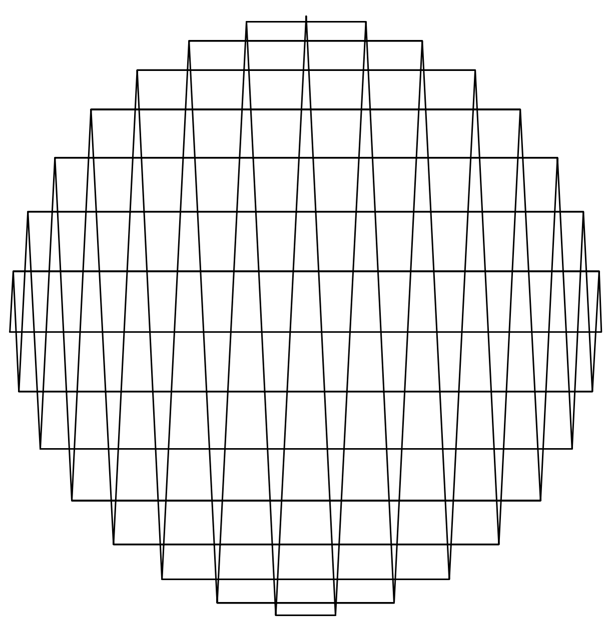
The only question that remains is: *What number is at the top of a general odd hill?* Recall that we defined the odd $\boldsymbol{n}$ as $\boldsymbol{n} = 2\boldsymbol{k}+1$ so that $\boldsymbol{k} = 1$, the first odd polygon, was a triangle connecting vertices 0-1-2. The largest sharpest triangle connected vertices 0-$\boldsymbol{k}$-($\boldsymbol{k}$+1)-0 and this triangle has $\boldsymbol{k}$ triangles if we allow overlapping counts. *The number of triangles in the final image is thus* $\boldsymbol{k}^2$. We can also answer this using $\boldsymbol{n}$ instead of $\boldsymbol{k}$ by solving for $\boldsymbol{k}$ from $\boldsymbol{n} = 2\boldsymbol{k}+1$. *The number of triangles in the final image is* $((\boldsymbol{n}-1)/2)^2$.

To test your understanding, what are the number of triangles in each of these four images? Answers are provided below.









There are various ways to approach these counts.

1. You could count vertices to find $\boldsymbol{n}$ then use the formula $((\boldsymbol{n}-1)/2)^2$

2. You could count vertices starting one vertex after the top (vertex 1) and going to just before half-way around since this is vertex $\boldsymbol{k}$, then note that the total is simply $\boldsymbol{k}^2$.

3. You could count the number of horizontal lines in the image (since each is a base for the triangle whose apex is at vertex 0. Of course, this is simply an alternative way to find $\boldsymbol{k}$ (the second method).

4. You could start at the side and visually see how many triangles have apexes just above the horizontal diameter (not shown). From here, add two for each vertex until you reach the top for reasons discussed here. (You can do this from either side: if you are starting in quadrant I, jump NW around the quarter circle; if starting in quadrant III, jump NE around the quarter circle). Then just square the count you have at the top, since that is $\boldsymbol{k}$.

Answers. Here are answers using the last method. Notice how the counts differ between even and odd ***k*** which are bolded in the counts below.

Upper Left: 2, 4, 6, **8** so there are 64 triangles.

Upper Right: 1, 3, 5, 7, 9, **11** so there are 121 triangles.

Lower Left: 2, 4, 6, 8, 10, 12, **14** so there are 196 triangles.

Lower Right: 1, 3, 5, 7, 9, 11, 13, **15** so there are 225 triangles.

Nonoverlapping Polygons

These Polygons are Triangles and Trapezoids

Where are they Located?

What happens as ***k*** increases?

Counting Nonoverlapping Polygons

Count by Row

Summing across Rows

Looking at Squares by Gnomons

Summing by Rows in a Different Way

One more Pattern, Gauss Addition

Optional: Visualizing added Vertices Two at a Time

If you just watch vertices get added (dots only) it looks like space is being made by moving everything around. This is especially true if you watch them get added one vertex at a time with vertex numbers attached. But if you think in terms of adding two lines at a time, and think in terms of quadrants, you see that there is one fixed point (the top) and two points at the bottom (k and k+1) that get closer to one another but stay on their side of the vertical diameter. All other vertices move toward the top (quadrants I and II) or bottom (quadrants III and IV) in order to make space for the two new vertices. If when the size was n, the closest vertices are in quadrants I and II, then the added points at n+2 will be in quadrants III and IV and *vice versa*. This is exactly why the waves are formed alternating between going up (longest horizontal line in I and II) and coming down (longest horizontal line in III and IV). All of this is most readily shown using the Excel file for sharpest triangles and focus your attention on the longest horizontal line.