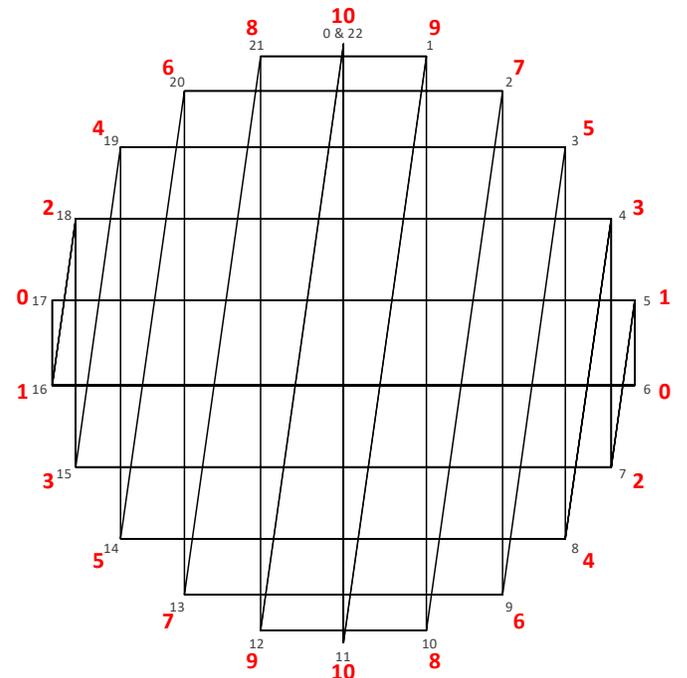
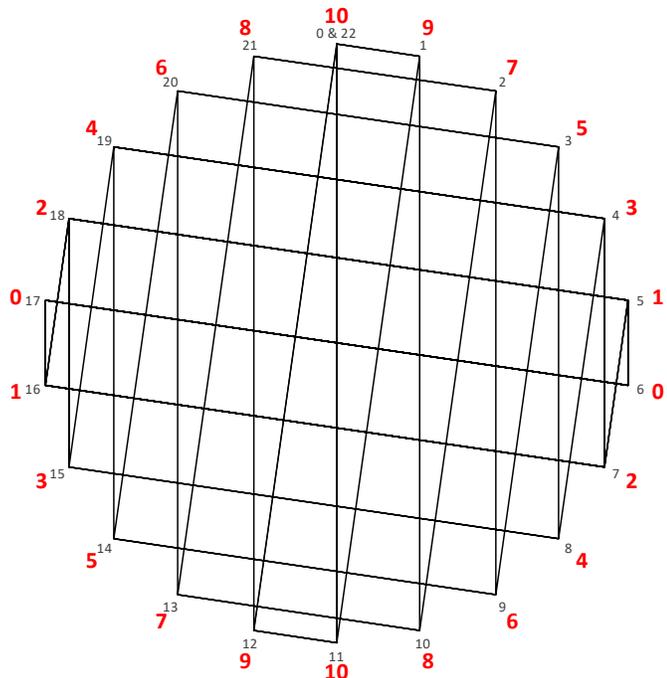
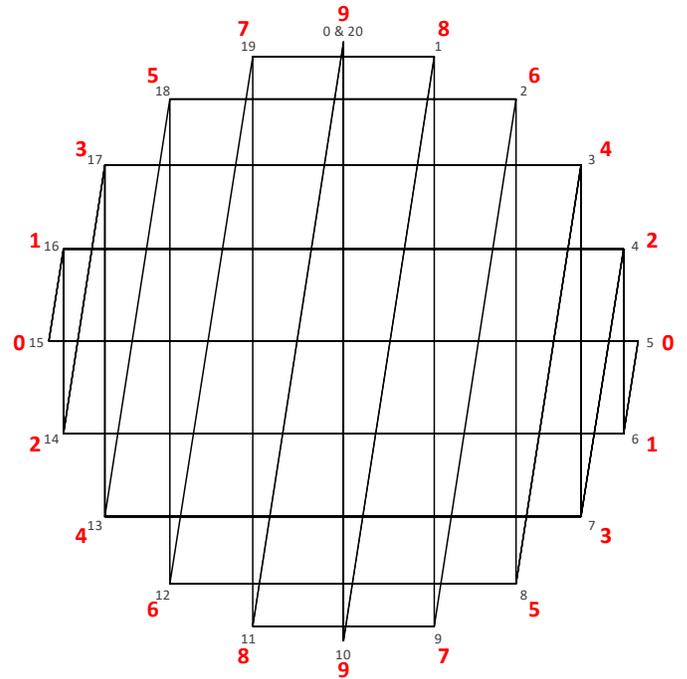
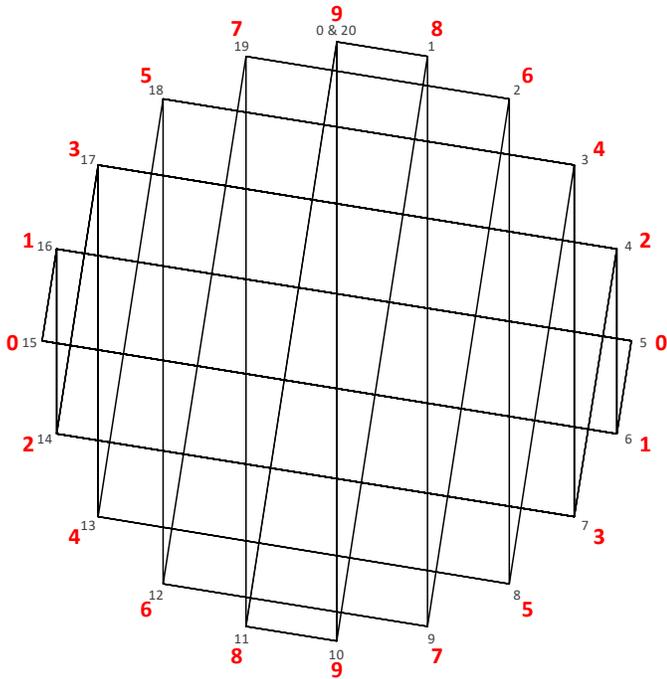


How Many Triangles are in the Sharpest Right Triangles Images?

The images below show $n = 2k+2$ for $k = 9$ and 10 , or $n = 20$ and $n = 22$, for *slanted* and horizontal orientations of sharpest right triangles images. These images include vertex labels in smaller black and **sharpest angle triangle counts in red**. Armed with what we learned in the last chapter, we see that things are different with right triangles than with isosceles. No longer is the pattern a drop of two as we move away from the top. Now the initial drop is one in the clockwise direction and two in the counterclockwise direction and after that there is a drop of two on both sides. The reverse pattern occurs looking upward from the bottom. Finally, notice that the same value occurs at the top and bottom of the vertical diameter. [These patterns depend, of course, on how we created the initial images. Had we created the images as mirrors (for example with sharp negatively sloped lines and vertical lines for legs with a positively sloped slanted bottom like 0-9-10 rather than 0-11-10 for upper left), these patterns would reverse.]



Just like with isosceles triangles, there is a better way to add **triangle counts** than going around the edge of the n -gon.

The Zig-Zag Pattern. The most parsimonious way to view what is happening is to look from side to side, much as we did in [counting isosceles triangles](#). We can look from left to right or right to left in each image but because we typically view vertices in a clockwise fashion, start at the right-most vertex which is not the sharpest angle for any triangle so sharpest angle count is **0** (but it is one of the bottom angles for a triangle). In both top row images this is at vertex $n/4 = 5$ and in both bottom row images it is at $(n+2)/4 = 6$. From here, follow the zig-zag path from vertex to vertex, counting as you go. In each instance, you increase sharpest angle triangle count by **1** for each zig or zag move until you get to the BOTTOM (at $n/2$). The count of triangles with sharpest angle at $n/2$ is **9** for the top row (count bottoms along slanted lines (for the *slanted* orientation) and horizontal lines (for the horizontal orientation) from vertices 1 to 9) and **10** for the bottom row. Next, confirm by looking at the four images that the same number of triangles have sharpest points at the TOP as bottom (**9** for $k = 9$ row and **10** for $k = 10$ row). (This always is the case because all four images exhibit 180° rotational symmetry.) Then continue zig-zag to the left half of each image, declining **1** for each zig or zag until you get to the left hand side of each image with a count of **0**. Both *slanted* and horizontal images have the same triangle counts at each of the vertices by row so that both images have the same *Total Number of Triangles*, $T(n)$.

Had we started at the left side, the only difference with the above discussion is that the first half of our zig-zagging would have taken us to the TOP, vertex 0, instead of the BOTTOM, vertex $n/2$.

We see the following sharpest angle triangle counts based on this zig-zag pattern (once we remove the **0**s at both ends):

Top: **1+2+3+4+5+6+7+8+9+9+8+7+6+5+4+3+2+1** Bottom: **1+2+3+4+5+6+7+8+9+10+10+9+8+7+6+5+4+3+2+1**

Both sums look like a hill with a flatter top (of two identical numbers rather than one number at the peak). So, one way we could answer is to use [The Hill Formula](#), $1 + 2 + \dots + (k-1) + k + (k-1) + \dots + 1 = k^2$, and add 9, or 10 to the answers.

The top row has $T(20) = 9^2 + 9 = 90$ triangles and the bottom row has $T(22) = 10^2 + 10 = 110$ triangles .

The number 20 and 22 relate to 9 and 10 because we defined n as $n = 2k+2$. Solving for k we have $k = (n-2)/2$.

Triangular Numbers. Another way to view this is via [triangular numbers](#), $\Delta_k = 1+2+\dots+k = k \cdot (k+1)/2$, discussed at the end of the last chapter.

For general k , the zig-zag pattern of numbers to sum is: $1 + 2 + \dots + (k-1) + k + k + (k-1) + \dots + 2 + 1$, where the two k s in the middle represent the counts at top and bottom of the vertical diameter.

This provides an even more natural way to look at triangular numbers since the second set of numbers is already there (from k to 1) and it is already organized from largest to smallest. Reorganizing the pair of sums from 1 to k by placing them on top of one another (moving around addition signs, making space adjustments between parts of the equation, adding parentheses, and adding notes to make the connections more obvious), we have:

$$\begin{array}{l}
 T(n = 2k+2) = \quad 1 + \quad 2 + \dots + (k-1) + k \quad \text{The right hand side of the first line of this equation is } \Delta_k . \\
 \quad \quad \quad + k \quad + (k-1) + \dots + \quad 2 \quad + 1 \quad \text{The second line is also } \Delta_k \text{ simply written in reverse order.} \\
 = (k+1) + (k+1) + \dots + (k+1) + (k+1) \quad \text{Add vertically, not horizontally. There are } k \text{ (k+1)s (see top row).}
 \end{array}$$

$$T(n = 2k+2) = k \cdot (k+1) = 2\Delta_k .$$

Given $k = (n-2)/2$, we could rewrite this (after simplifying) as: $T(n) = n \cdot (n-2)/4$.

As a check of this last equation, note that $T(20) = 20 \cdot 18/4 = 10 \cdot 9 = 90$ and $T(22) = 22 \cdot 20/4 = 11 \cdot 10 = 110$ the same numbers we obtained by adding k to *The Hill Formula* above.