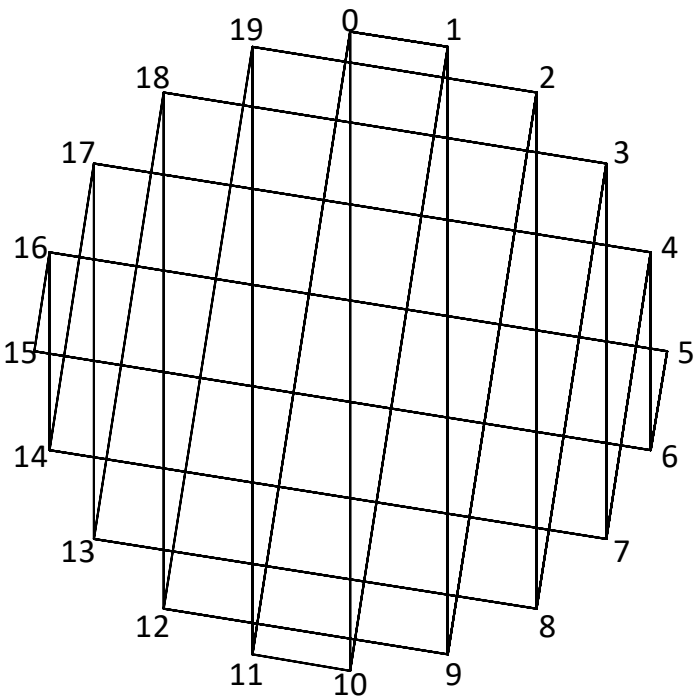
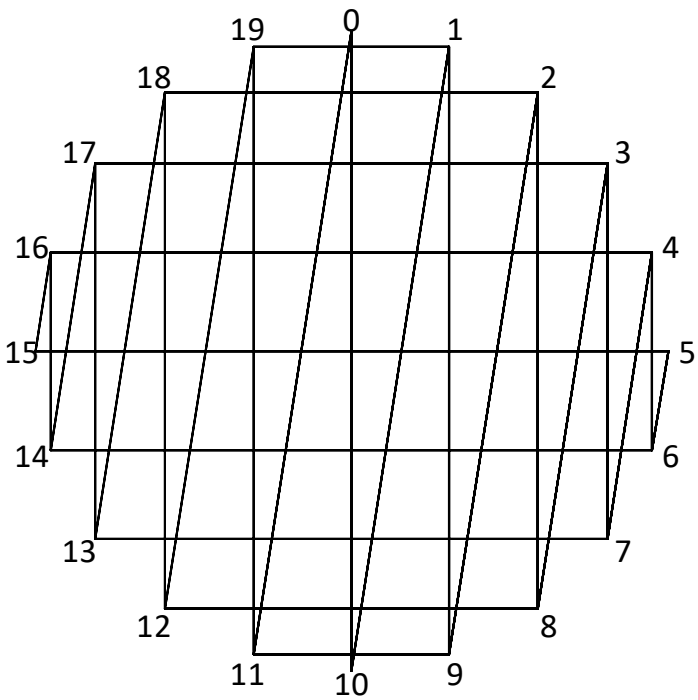
Sharpest Right Triangles on Even Polygons

If we start with an even ***n***-gon, then it is worthwhile to define ***n*** as ***n*** = 2***k***+2 so that ***n*** represents the ***k***th even polygon, the first being a square, ***n*** = 4 if ***k*** = 1. The sharpest angle spans a single vertex in a sharpest triangles image. The remaining angles must cumulatively span an odd number of vertices, 2***k***+1 = ***n***-1, since the total span of the angles of a triangle is ***n*** vertices, see E2.6.3. To have the other two angles produce as close to a sharpest apex isosceles triangle as possible requires 1, ***k***, and ***k***+1 as the number of vertices spanning each side. The three angles from smallest to largest are 180/***n***° = 180/(2***k***+2)°, 180***k***/(2***k***+2)°, and 180·(***k***+1)/(2***k***+2)°. ***NOTE.*** We call the side opposite the sharpest angle the *bottom* since *base* is reserved for isosceles triangles and the sharpest angle is not the *apex* since that also refers to isosceles triangles.

Why Right Angles? The largest of these angles is a right angle because it spans exactly half of the ***n***-gon's vertices, ***n***/2 = ***k***+1, so 90° = (180·(***k***+1)/(2***k***+2))°. ***NOTE:*** It is IMPOSSIBLE to obtain right angles using vertices of an odd ***n***-gon.

Two Configurations. *There are two alternative configurations of sharpest triangles for even **n** images having these angles. One configuration uses all vertices of the **n**-gon for larger angles (bottoms) and all but two for sharpest angles. The other omits a pair of opposing vertices from having larger angles.* Since this sounds abstract, consider these ***n*** = 20, ***k*** = 9 images. Two of three directions are the same in both images. Both include the vertical diameter of the ***n***-gon going from 0-10 (0-***n***/2) and other vertical lines and both include the positively-sloped line from 0-11 (0-(***n***/2+1)) and other lines parallel to 0-11. The difference is the third direction (which produces the bottoms). **Call the first *slanted*, and the second <u>horizontal</u>.** Notice that the *slanted* image has three lines at each vertex except for two lines at vertices 5 and 15. By contrast, the <u>horizontal</u> has two additional vertices with only two lines, 0 and 10 both of which form the sharpest angle. *Vertices 0 and 10 (**n**/2) are not used for bottoms of other triangles, just as described in italics above.*





The *slanted* image has vertices that are more readily discernable since all three vertices of the largest triangle are polygonal vertices. There are two such largest triangles, 0-10-11 and 0-1-10, both have hypotenuse 0-10 with right angled vertices at 11 and 1. All triangles *except* these two have one, or at most two, vertices on the polygon. The other triangles are at the intersection of the additional lines that are parallel to the three sides of 0-10-11. Each triangle has a right angle that is formed at the intersection of the two slanted lines. The easiest right angles are to see are at vertices 5 and 15 since there are only two lines at each. Both angles span from 16-6 = 10 = ***n***/2 vertices and hence are right angles.

The right angles in the <u>horizontal</u> image (at right) are more readily visible because two of the sets of perpendicular parallel lines are vertical and horizontal. In this case, the largest hypotenuses are from 0-11 and 1-10 with right angles just beneath 0 or just above 10 at the intersections of the lines 1-19 or 9-11, with the 0-10 vertical diameter.

Two Configurations, Different Sized Triangles. The triangles in both images are similar (meaning equal angles) but they are not identical. Identical triangles have equal angles and equal sides. **It turns out that NONE of the triangles in one image are identical to ANY of the triangles in the other.**

Consider the triangles using vertices 5 and 6. The length 5-6 is the same in both images and in both it is a positively sloped line (+ in the table). In the *slanted* version, it is the longer leg of the right triangle but in the horizontal version it is the hypotenuse. Since the hypotenuse is always longer than either leg in a right triangle, the triangle using 5-6 is larger in the *slanted* version than the horizontal. The reverse is true for the next line over, the vertical line 4-6 (V in the table). The table shows what happens for every + and V line in the image.

Comparison of Largest Triangle Using Specfic Lines as either the *Long Leg* **L** or *Hypotenuse* **H** across Images

Vertical **V** vs. **+** Slope	Line	Left *Slanted*	Right Horizontal	Larger Triangle?
+	**5 - 6**	**L**	**H**	*Slanted*
V	**4 - 6**	**H**	**L**	Horizontal
+	**4 - 7**	**L**	**H**	*Slanted*
V	**3 - 7**	**H**	**L**	Horizontal
+	**3 - 8**	**L**	**H**	*Slanted*
V	**2 - 8**	**H**	**L**	Horizontal
+	**2 - 9**	**L**	**H**	*Slanted*
V	**1 - 9**	**H**	**L**	Horizontal
+	**1 - 10**	**L**	**H**	*Slanted*
V	**0 - 10**	**H**	*	***Slanted*** **×2**
+	**0 - 11**	**L**	**H**	*Slanted*
V	**19 - 11**	**H**	**L**	Horizontal
+	**19 - 12**	**L**	**H**	*Slanted*
V	**18 - 12**	**H**	**L**	Horizontal
+	**18 - 13**	**L**	**H**	*Slanted*
V	**17 - 13**	**H**	**L**	Horizontal
+	**17 - 14**	**L**	**H**	*Slanted*
V	**16 - 14**	**H**	**L**	Horizontal
+	**16 - 15**	**L**	**H**	*Slanted*
* Does not use all of 0-10 as a side.				

The pattern alternates between *slanted* and horizontal being largest EXCEPT for the vertical diameter. As noted above, the vertical diameter is not used as a base in the horizontal configuration, 0-10 in the figures and table. As a result, it cannot be largest. Note also that the vertical diameter is hypotenuse to two triangles, not one as is the case for all other + and V lines. Those triangles are 0-10-11 and 0-1-10 and both are the largest triangles in the image.

If we choose a different even ***n***, then, of course, the table will need to be modified but the same basic structure remains. When ***n*** is divisible by 4, like ***n*** = 20, the shortest right side line (first row of the table) with be + connecting ***n***/4 to ***n***/4+1. If ***n*** is divisible by 2 but not 4 then the shortest right side line will be V connecting (***n***-2)/4 to (***n***+2)/4, for example the line 2-3 is vertical for ***n*** = 10 (shown below). The horizontal triangle with right angle at 3 is larger than the *slanted* triangle with 72° angle at 3.

A bit of introspection provides a rationale (beyond the distinction between leg and hypotenuse) for the alternating difference in size of these triangles. That rationale has to do with considering the location of the third leg which we are calling the bottom. The image at right starts with a horizontal ***n*** = 10 image and adds **5 *slanted* bottom lines in red** so that both configurations are seen on the same image. When the line is vertical but not the vertical centerline (like 2-3, 1-4, 9-6 or 8-7), the horizontal configuration has a small triangle added at the bottom (moving from *slanted* to horizontal) noted by **H** inside that triangle except near 3 and 8 making it the larger triangle. When the line is positively sloped (like 2-4, 1-5, 0-6 or 9-7) the slanted configuration has a small triangle added at the bottom (in moving from horizontal to *slanted*) noted by ***S*** inside that triangle making *slanted* the larger triangle.