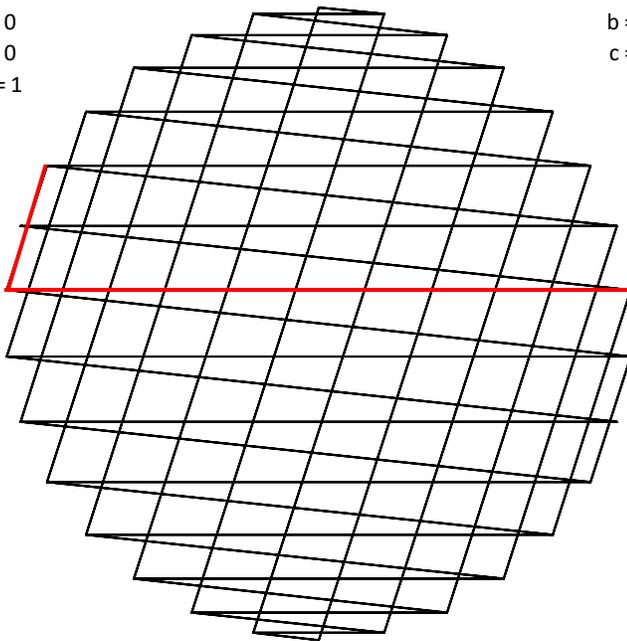# Counting General Sharpest Triangles by Focusing on Angles

The discussion thus far has focused on specific types of sharpest triangles: isosceles, right, least obtuse, most obtuse, and so on. By doing this we were able to derive equations that work as *n* varies for that specific type of triangle. Here we examine the opposite side of the coin, so to speak.

Suppose we are confronted with images that are quite clearly sharpest but are not one of the types listed above. Further, they have been drawn with sharpest angles that move roughly "sideways" rather than "up and down." The two images below have their attributes listed to the left but our purpose here is to explain the *a*, *b*, and *c* values listed to the right. These are whole numbers which sum to *n* and represent the arcs of a circle that describe angles of triangular images created using polygonal vertices. Since the triangles are sharpest, one of the three numbers, *a*, must be 1 (see here for additional detail). We can talk about *a*, *b*, and *c* as angles knowing that they must be multiplied by 180/*n*° to obtain the degree values. Unless *b* and *c* are the same size, we can talk about one as being medium and the other as large.
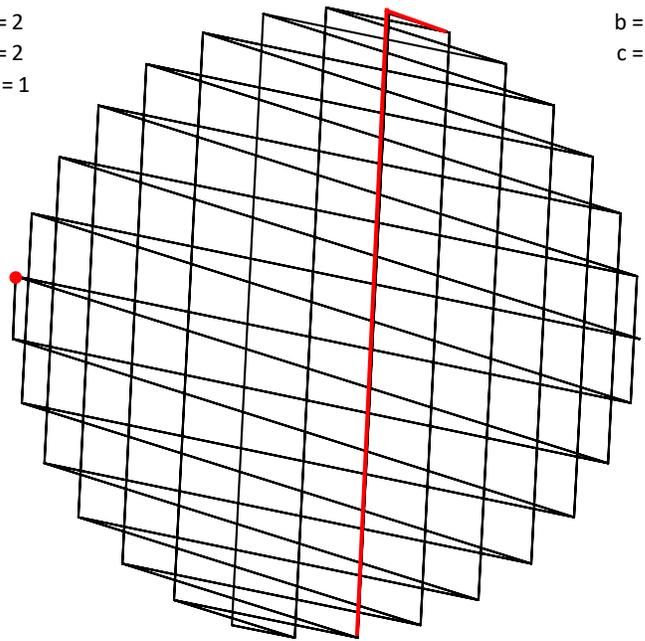
Focus on the Medium-Sized Angle, *m*. We know that the smallest angle spans 1 vertex and we know that the sum is *n*, so if we find the size of the medium-sized angle, *m*, we can easily find the size of the larger one as *L* = *n*-*m*-1. This is most easily obtained by using inscribed angles rather than interior angles since then you only need count the vertices between angle endpoints on one side rather than both sides as shown below. Additionally, it helps if you can find a vertex where the medium-sized angle is *isolated* meaning that there are only two lines at that vertex.

Right Image. The right image has a medium angle isolated at **vertex 1** with **red lines** superimposed to show the angle created at that vertex. A quick count shows that this angle spans 13 vertices (the angle is created by connecting **15-1-2**). (Vertex numbers have been suppressed in both images to encourage you to manually count between angle endpoints.)

Left column (left image):
j = 18   n = 30
k = 0
v = 0
w = 1

Middle column (right image):
a = 1   j = 16   n = 31
b = 17   k = 2
c = 12   v = 2
        w = 1

Right column (right image):
a = 1
b = 13
c = 17

Left Image. Notice that one cannot isolate the medium-sized angle in the left image. The only vertices having two lines that are not points of the sharpest angle (at vertices 9 and 24) are at the top and bottom and both are largest angles. The **red line overlays** for the middle-sized angle at vertex **23** creates the angle **7-23-25**. A quick count from the vertex **7** counterclockwise to **25** makes two things clear: **1)** the angle spans 12 vertices; **2)** 12 is also the count of triangles whose peak is vertex **7** since each of those counterclockwise counted vertices has a line parallel to **23-25** that forms a base for the triangles with vertex **7** as peak. This value (12) is the height of the plateau of triangles counts for the left image.

We could have counted vertices for the right image using a vertex with 3 lines that shows the number of triangles at the plateau instead of using vertex **1** since it had only two lines. For example, the medium-sized angle at **red dotted vertex 24** which is composed of 10-**24**-23 (the single vertex jump leg on the left-hand side of the right image is 23-**24**) also spans

13 vertices and each of the vertices 11, 12, …, 23 has a line parallel to 23-**24** that acts as a base for a sharpest triangle whose peak is at vertex 10. Put another way, 13 is the height of the plateau value of triangles counts for the right image.

The larger angle is readily obtained as $L = n-m-1$. In both images, this is 17 and is noted by **b** in the left and **c** in the right image. Although both larger angles spanned 17 vertices, the angles vary across images because **n** varies. In particular, the large angle for the **n** = 30 left image is 102° = 180·17/30°, while for the **n** = 31 right image it is 98.7° = 180·17/31°.

The angle values are shown as **a**, **b**, and **c** in the *Excel* file and the manipulations necessary to get these values is shown in the blue area there. By construction, **a** is smallest so **a** = 1 for sharpest triangles but **b** and **c** may be in either order as these examples have shown. For our present purpose, it is best to call these angles **m** and **L** for *medium* and *large*.

Counting Triangles. Here we are interested in the number of triangles total, not in how those totals relate to truncated isosceles or truncated right triangle versions we examined in the last section. To examine this, we focus on how the angles help us answer this question. To simplify counting we use stacked squares to represent triangles counts.

We know that triangle counts start at 0 on each side and increase by 1 as we follow a zig-zag path until that count reaches a plateau. We just argued that the plateau is the size of the medium angle, **m**. We know that there are **m** vertices with triangles counts less than **m** on each side or 2**m** total. Add to that a single vertex at triangles count of **m** and we have m^2 total triangles obtained from these 2**m**+1 vertices using *The Hill Formula* with the rest of the vertices at the plateau value of **m**. Thus, the total number of triangles as a function of **n** given a medium angle of **m**, T(**n**; **m**), is:

$$T(n; m) = m^2 + m(n-(2m+1)) = m(m + (n-(2m+1))) = m(m + n-2m-1) = m(n-m-1) = m \cdot L .$$

The rationale for this striking result is depicted in the graphic below, based on triangles counts from the left and right images on the prior page. *In contrast with our earlier discussion, this single formula that covers both even and odd n.*

The green isosceles right triangle is **m** wide and high. By rotating the triangle 90° clockwise and placing the green cell 1 in the starting cell 0 we obtain an **m**×**m** square together with a **m**×(**n**-(2**m**+1)) rectangle. Rather than focus on the individual parts, notice that the total is a rectangle that is **m** high and **n**-**m**-1 = **L** long. This means there are a total of **m**·**L** triangles total, or 204 on the left and 221 on the right. Of course, the other formulas still apply. For example, note that the **n** = 31 right image has 15^2-2^2 = 221 triangles, since the deviation **d** from isosceles **k** = 15, is **d** = 2: **k**-2 = 13 = **m** and **k**+2 = 17 = **L**.

Figure: Triangles counts by zig-zag vertex for each image. "Vertex 23 is noted in red, see 7-23-25 discussion." / "Rotate the green isosceles right triangle clockwise 90° and move into place to create a rectangle as below." / "Tying Triangles Counts by Vertex to Angle Sizes m and L. Vertex 24 is noted in red, see 10-24-23 discussion."

Left image: This is the size of the largest angle L = n-m-1. This is 12 wide & high, middle ∠m.
30 = n; 12 = m, middle angle span; 1 = sharpest angle; 17 = L, largest angle = n-m-1.
The resulting rectangle has 12 × 17 = 204 squares. Therefore, the left image has 204 triangles total.

Right image: This is the size of the largest angle L = n-m-1. This is 13 wide & high, middle ∠m.
31 = n; 13 = m, middle angle span; 1 = sharpest angle; 17 = L, largest angle = n-m-1.
The resulting rectangle has 13 × 17 = 221 squares. Therefore, the right image has 221 triangles total.