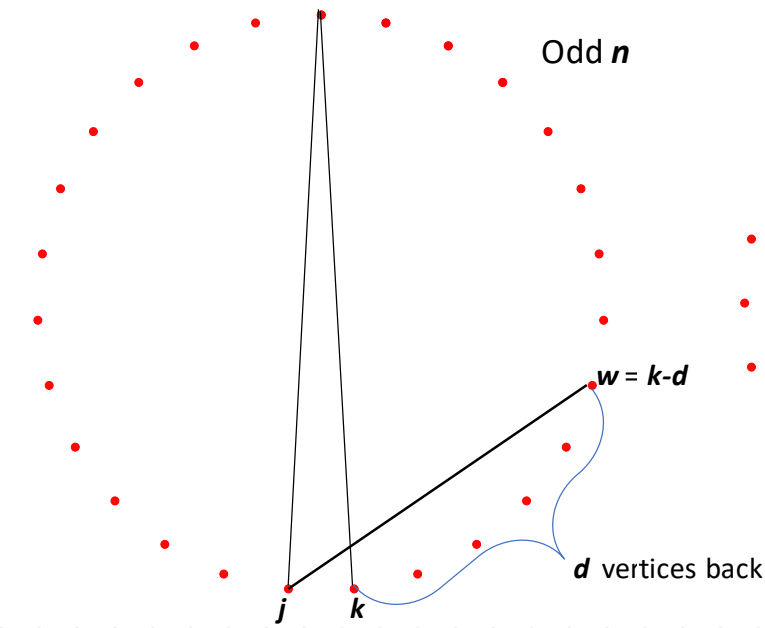
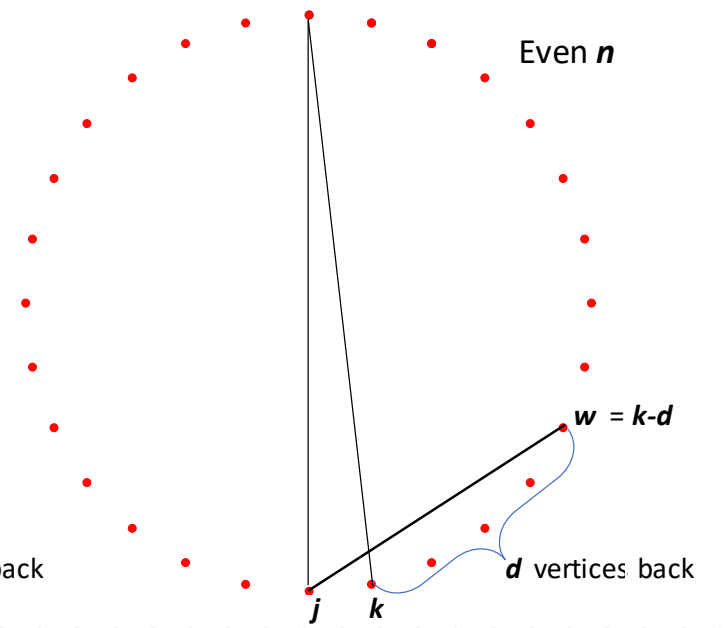
General Sharpest Triangles as Deviations from Sharpest Isosceles or Right Triangles

We have now examined all sharpest triangles that are largest, second largest, smallest and second smallest using the rubric laid out a couple of sections ago. Using the layout suggested there, let j = INTEGER((n+1)/2) and $k = j$-1. This creates largest triangle straddling the bottom for odd n and with j being the vertical diameter for even n. Additionally, let $v = j$. We can describe all possible largest triangles using the location of the line from v to w.

Let d be the deviation from k, or the number of vertices BEHIND k. Set $w = k-d$. The bounds on d are $0 \le d \le k-1$. If $d = 0$, we have sharpest isosceles or sharpest right triangles depending on whether n is odd or even. When $d = 1$ we have the sharpest least obtuse scalene case we just examined. When $d = k-1$ we have sharpest most obtuse isosceles triangles since $w = 1$ and when $d = k-2$ we have sharpest most obtuse scalene triangles since in that case $w = 2$.

Consider the situation for an indeterminate d somewhere in the middle of the range of d values. The situation is shown for odd and even n in the two images below (of course, you could count dots to figure out n but just think in general terms of odd and even n. This sets up a plateau of maximum triangle counts from a vertex of $w = k-d$. The reasoning here is simple: Consider the largest triangle created in this situation. Each vertex from 1 to w will have a line parallel to line j-w that intersects the largest triangle thereby creating a count of w triangles at vertex 0. This is the maximum triangle count.





How many vertices have a maximum triangle count in this situation? We know that triangle counts increase by 1 on each side starting at 0 until they get to w. This means that there are w vertices on each side with counts of less than w, or $2w$ on both sides. Add a single w to this and we have 0, 1, 2, …, w-1, w, w-1, … 2, 1, 0 from $2w$+1 vertices. The sum of these is w^2 by *The Hill Formula*. The rest of the vertices, n-($2w$+1), have a count of $w = k-d$ triangles per vertex.

Odd $n = 2k+1$. $T(n) = w^2+w(n-(2w+1)) = w^2+w((2k+1)-(2w+1)) = w^2+2w(k-w) = (k-d)^2+2(k-d)(d) = (k^2-2dk+d^2) + (2dk-2d^2)$

$$T(n) = k^2 - d^2$$

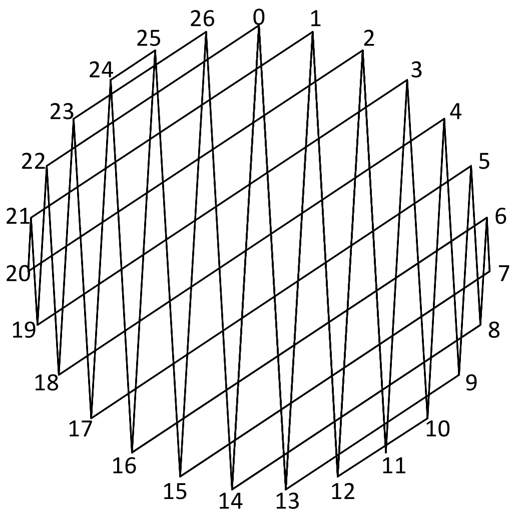
Thus, for odd $n = 2k+1$, if the deviation from k is d, the total number of triangles is k^2-d^2. If you have an odd n-gon sharpest triangles image and see a plateau of $w = k-d$ maximum triangle counts, then the total number of triangles in the image is $k^2-(k-w)^2$ where $k = (n-1)/2$, or number of isosceles triangles from an n-gon less those from a ($2d$+1)-gon.

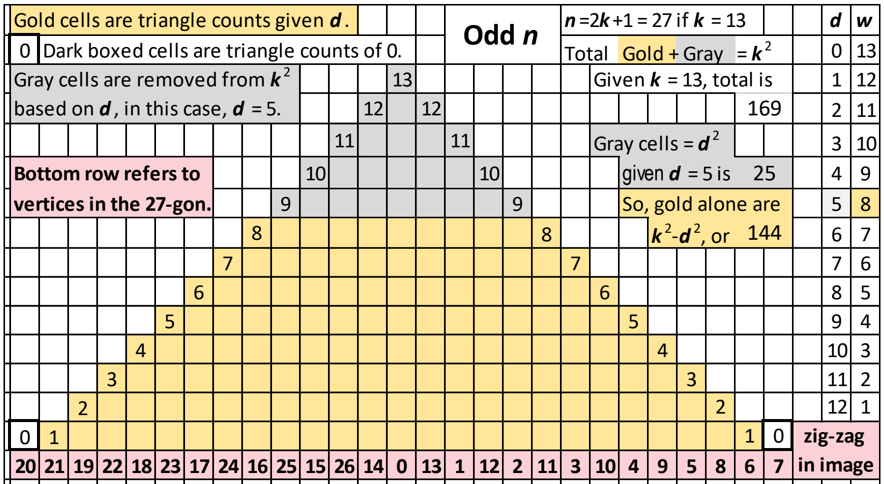
Even $n = 2k+2$. $T(n) = w^2+w(n-(2w+1)) = w^2+w((2k+2)-(2w+1)) = w^2+w(2k-2w+1) = (k-d)^2+(k-d)(2d+1)$

$$= (k^2-2dk+d^2) + (2dk-2d^2+k-d) = (k^2+d^2) + (-2d^2+k-d) = k^2+k-(d^2+d) = k(k+1) - d(d+1)$$

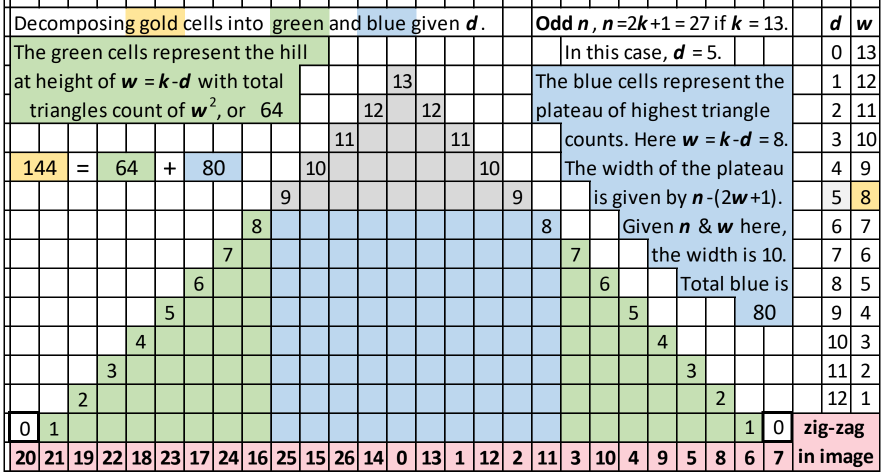
The total number of triangles for even n is the total right triangles less the total right triangles from a ($2d$+2)-gon.

Both answers are more readily understood by considering the zig-zag progression of counts of triangles using blocks.





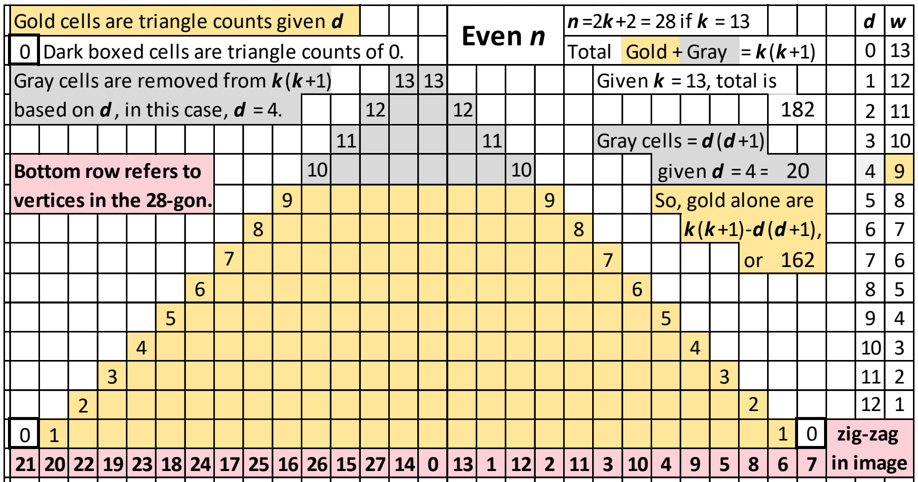
Gold cells are triangle counts given *d*.															**Odd *n***				$n=2k+1=27$ if $k=13$								*d*	*w*
$\boxed{0}$	Dark boxed cells are triangle counts of 0.																		Total Gold + Gray $=k^2$								0	13
Gray cells are removed from k^2													13							Given $k=13$, total is							1	12
based on *d*, in this case, $d=5$.												12		12											169		2	11
											11				11					Gray cells $=d^2$							3	10
Bottom row refers to										10						10					given $d=5$ is				25		4	9
vertices in the 27-gon.									9								9				So, gold alone are						5	8
								8										8				k^2-d^2, or			144		6	7
							7												7								7	6
						6														6							8	5
					5																5						9	4
				4																		4					10	3
			3																				3				11	2
		2																						2			12	1
$\boxed{0}$	1																								1	$\boxed{0}$	**zig-zag**	
20	**21**	**19**	**22**	**18**	**23**	**17**	**24**	**16**	**25**	**15**	**26**	**14**	**0**	**13**	**1**	**12**	**2**	**11**	**3**	**10**	**4**	**9**	**5**	**8**	**6**	**7**	**in image**	



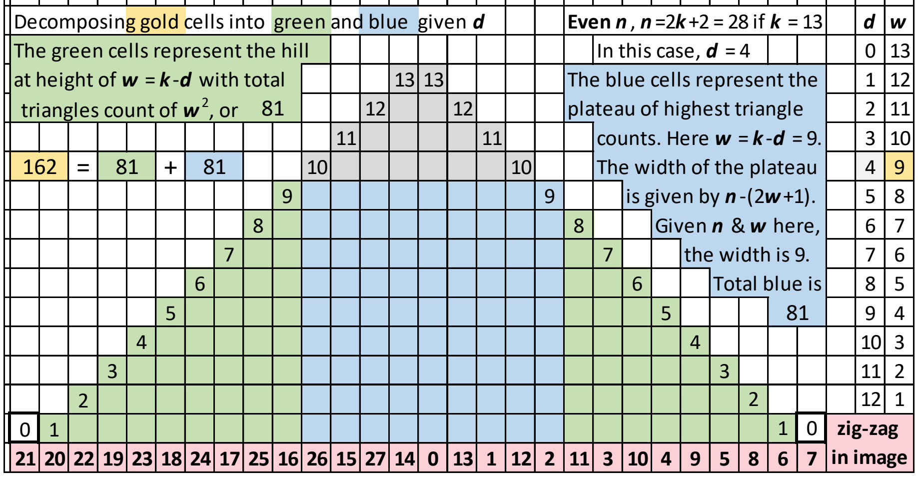
Decomposing gold cells into green and blue given *d*.																		**Odd *n***, $n=2k+1=27$ if $k=13$.									*d*	*w*
The green cells represent the hill																			In this case, $d=5$.								0	13
at height of $w=k-d$ with total													13					The blue cells represent the									1	12
triangles count of w^2, or									64			12		12				plateau of highest triangle									2	11
											11				11				counts. Here $w=k-d=8$.								3	10
144		=	64		+	80				10						10			The width of the plateau								4	9
									9								9			is given by $n-(2w+1)$.							5	8
								8										8			Given n & w here,						6	7
							7												7			the width is 10.					7	6
						6														6			Total blue is				8	5
					5																5				80		9	4
				4																		4					10	3
			3																				3				11	2
		2																						2			12	1
$\boxed{0}$	1																								1	$\boxed{0}$	**zig-zag**	
20	**21**	**19**	**22**	**18**	**23**	**17**	**24**	**16**	**25**	**15**	**26**	**14**	**0**	**13**	**1**	**12**	**2**	**11**	**3**	**10**	**4**	**9**	**5**	**8**	**6**	**7**	**in image**	

The odd n shown above was $n=27$ so $k=13$ and $d=5$. The completed image has its third line from j-w = 14-8 since $j=k+1$ and $w=k-5$. Counting zig-zag from vertex 7 we see the first of 11 vertices with triangles count of 8 is at vertex 11 and the last is 16 (see pink highlighted vertex numbers).

The difference between squares T = k^2-d^2 is clearer if we note that each square is a hill of numbers from *The Hill Formula*, so the difference simply removes the top of the hill.



Gold cells are triangle counts given *d*														**Even *n***					$n=2k+2=28$ if $k=13$									*d*	*w*
$\boxed{0}$	Dark boxed cells are triangle counts of 0.																		Total Gold + Gray $=k(k+1)$									0	13
Gray cells are removed from $k(k+1)$													13	13						Given $k=13$, total is								1	12
based on *d*, in this case, $d=4$.												12			12											182		2	11
											11					11				Gray cells $=d(d+1)$								3	10
Bottom row refers to										10							10					given $d=4=$				20		4	9
vertices in the 28-gon.									9									9				So, gold alone are						5	8
								8											8				$k(k+1)-d(d+1)$,					6	7
							7													7					or	162		7	6
						6															6							8	5
					5																	5						9	4
				4																			4					10	3
			3																					3				11	2
		2																							2			12	1
$\boxed{0}$	1																									1	$\boxed{0}$	**zig-zag**	
21	**20**	**22**	**19**	**23**	**18**	**24**	**17**	**25**	**16**	**26**	**15**	**27**	**14**	**0**	**13**	**1**	**12**	**2**	**11**	**3**	**10**	**4**	**9**	**5**	**8**	**6**	**7**	**in image**	



Decomposing gold cells into green and blue given *d*																			**Even *n***, $n=2k+2=28$ if $k=13$									*d*	*w*
The green cells represent the hill																				In this case, $d=4$								0	13
at height of $w=k-d$ with total													13	13					The blue cells represent the									1	12
triangles count of w^2, or									81			12			12				plateau of highest triangle									2	11
											11					11				counts. Here $w=k-d=9$.								3	10
162		=	81		+	81				10							10			The width of the plateau								4	9
									9									9			is given by $n-(2w+1)$.							5	8
								8											8			Given n & w here,						6	7
							7													7			the width is 9.					7	6
						6															6			Total blue is				8	5
					5																	5				81		9	4
				4																			4					10	3
			3																					3				11	2
		2																							2			12	1
$\boxed{0}$	1																									1	$\boxed{0}$	**zig-zag**	
21	**20**	**22**	**19**	**23**	**18**	**24**	**17**	**25**	**16**	**26**	**15**	**27**	**14**	**0**	**13**	**1**	**12**	**2**	**11**	**3**	**10**	**4**	**9**	**5**	**8**	**6**	**7**	**in image**	

The even n shown was $n=28$ so $k=13$ with $d=4$. The completed image has its third line from j-w = 14-9 since $j=k+1$ and $w=k-4$. The first triangles count of 9 is vertex 2 and the last is vertex 16.

As with odd n, the total number of triangles can be seen as a difference between the n-gon and the (2d+2)-gon totals since the latter simply removes the top d rows of the hill. The difference is the top is 2-wide for even n, but 1 for odd n.