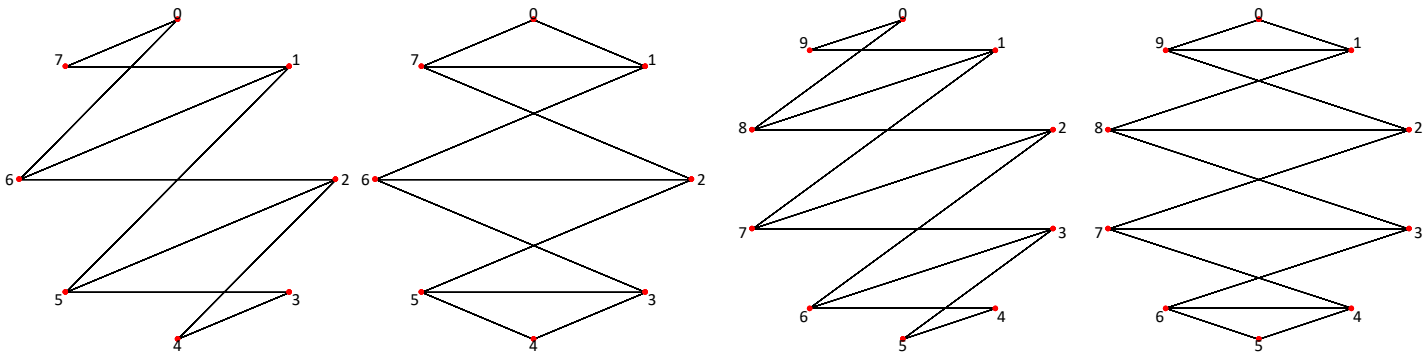
Sharpest Most Obtuse Isosceles and Scalene Triangles Images

Sharpest angle triangles occur if one of the angles spans a single vertex. If the other two angles are as close to one another as possible we obtain sharpest isosceles triangles if ***n*** is odd and sharpest right triangles if ***n*** is even. As noted in the previous section, all other types of sharpest triangles are obtuse.

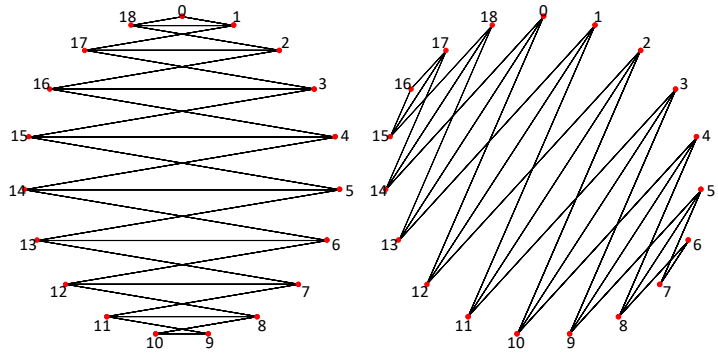
Among sharpest obtuse triangles, the most obtuse occurs if its largest angle spans as many vertices as possible. When this happens, we have a *smallest large triangles image* as noted here with isosceles triangles whose base angles span a single vertex.

Sharpest Most Obtuse Isosceles. If both base angles span a single vertex, then the image is the most obtuse isosceles image with angles that span (1,1,***n***-2). As we saw there for ***n*** = 18, there are two versions of such triangles when ***n*** is even (***n*** = 8 and 10 are shown below) but only one when ***n*** is odd. The reason for this distinction is that when ***n*** is even, one can always produce images with two opposing smallest triangles whose bases span a single vertex and has an interior apex (like the first and third images with bases at ***n***-1 to 0 and ***n***/2 to ***n***/2-1) or four smallest triangles with bases that span two vertices two of which have apexes on the ***n***-gon (at 0 and ***n***/2 in the second and fourth images).

From these examples we can see that the largest triangles vary depending on whether ***n*** = 4***k*** or ***n*** = 4***k***+2. The left set represents ***n*** divisible by 4 and in this instance (for ***k*** ≥ 2), there are four largest triangles each of which has a base that spans ***n***/2-1 vertices (like 1-6 and 2-5 in the ***n*** = 8 first image) and two largest triangles, each of which has a base that spans ***n***/2 vertices and is a diameter of the ***n***-gon (like 2-6 in the ***n*** = 8 second image). When ***n*** is divisible by 2 but not 4 like the right set (and ***n*** = 18 seen earlier), the situation reverses itself. The version with smallest triangles spanning a single vertex also contains two largest triangles that span a diameter (like 2-7 in the ***n*** = 10 third image) and the version with four smallest triangles spanning two vertices two of whom have apexes on the ***n***-gon, have four largest triangles each having a base spanning ***n***/2-1 vertices (like 2-8 and 3-7 in the ***n*** = 10 fourth image).



By contrast, when ***n*** is odd, like these two ***n*** = 19 images, one end of the image will have a smallest triangle with base spanning two vertices like 1 to ***n***-1 with apex at 0 on one end at left with a base spanning a single vertex on the other end from (***n***-1)/2 to (***n***+1)/2. The apex that coincides with an odd ***n***-gon vertex need not be at 0. Once its location is set at vertex ***v***, the location of the single span base is at ***v***+(***n***-1)/2 to ***v***+(***n***+1)/2 if ***v*** < ***n***/2. If ***v*** > ***n***/2, the single span base is from ***v***-(***n***-1)/2 to ***v***-(***n***+1)/2. The right ***n*** = 19 image has ***v*** = 16 so the single span base is from 16-(19-1)/2 = 7 to 16-(19+1)/2 = 6. For odd ***n***, there are always two largest triangles whose base spans (***n***-1)/2 vertices with one end pointed, and the other flat. For even ***n*** images, either both ends are flat (like first and third) or both are pointed (like second and fourth) AND both versions are possible for each even ***n***.

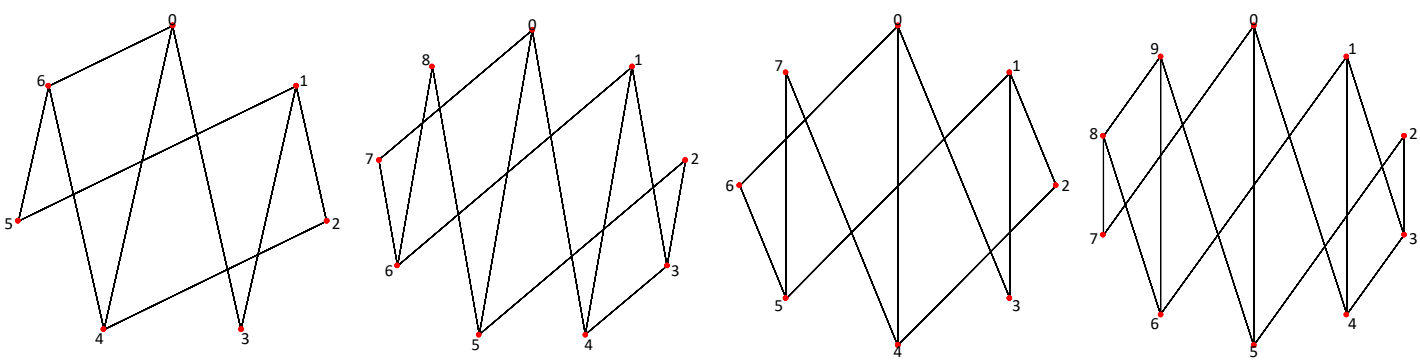


Counting Sharpest Most Obtuse Isosceles Triangles. Sharpest most obtuse isosceles triangles have two sharpest base angles but only one most obtuse apex angle of 180·(***n***-2)/***n***°. We could continue to count sharpest angles using the vertices of the ***n***-gon, but we would have to divide our answer by 2 to avoid double counting. A better solution is to

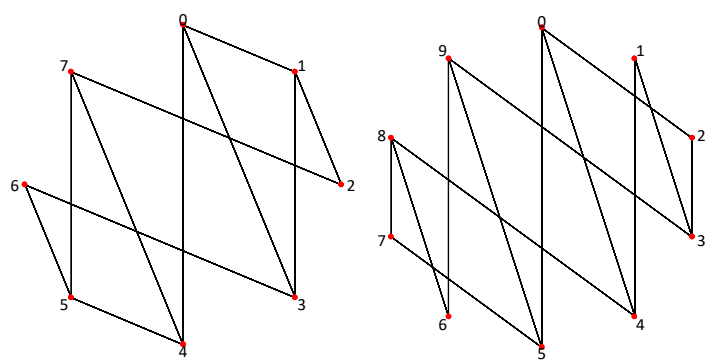
define the apex of the isosceles triangle as the *distinguished point* and count apexes. Most of those apexes are interior to the ***n***-gon and all are on the diameter bisecting the bases of these isosceles triangles. Note that all the interior apexes create pairs of triangles while apexes on the ***n***-gon have only one triangle at that apex. Both points lead to the conclusion that whether ***n*** is even or odd, there are ***n***-2 sharpest most obtuse isosceles triangles in the image.

Perhaps the easiest to see are the first and third even ***n*** images because you can think of the bands parallel to 0-(***n***-1). The bottom of the first band starts at vertex 1 and the bottom of the last band starts at vertex ***n***/2-1 (3 in the first image, 4 in the third) meaning that there are ***n***/2-1 interior apexes each having two opposing triangles or 2·(***n***/2-1) = ***n***-2 triangles in these images. The second and fourth have ***n***/2-2 interior apexes and 2 apexes on the ***n***-gon or 2·(***n***/2-2)+2 = ***n***-2 total per image. The odd images have (***n***-1)/2-1 interior apexes (on horizontal bands with bottoms starting at vertices 2 to 9 given the ***n*** = 19 left image plus a single apex at vertex 0), so the total number of triangles is 2·((***n***-1)/2-1)+1 = ***n***-2 triangles for odd ***n***. All three versions (single vertex jump smallest triangles base and double vertex jump smallest triangles base at both ends for even ***n***, and one of each for odd ***n***) produce the same result.

Sharpest Most Obtuse Scalene Triangles. When the largest triangle is second smallest, then we obtain the most obtuse scalene triangles image with angles that span (1,2,***n***-3). The largest angle will be obtuse if (***n***-3)/***n*** > ½ which means ***n*** > 6. The four images below show the progression obtained using the *Excel* file for creating general triangular images having jump ***j*** be ***j*** = INTEGER((***n***-1)/2), ***k*** = ***j***+1, and ***v*** = ***k***, and ***w*** = 2. This produces images which have an angle spanning a single vertex at 0 (angle ***j***0***k***) and two vertices at vertex ***k*** (angle 0***k***2). Versions of this for ***n*** = 7, 9, 8 and 10 are shown from left to right. When ***n*** is odd, like the first and second images, the largest triangle in the image has its single vertex span at the bottom of the image just like the odd sharpest isosceles triangles from the last chapter, but the bottom no longer spans across the base. Instead it connects to vertex 2 and the angle thus created spans two vertices, and this creates the second smallest large triangle. When ***n*** is even, like the third and fourth, there is a vertical line from 0 to ***n***/2 and another line from 0 to ***n***/2-1 with third line from ***n***/2 to 2 just like the second smallest large triangle shown earlier for ***n*** = 18.



*Two versions for even **n**, one for odd **n**.* Just as we have seen before, there are two versions for even ***n*** and one for odd ***n***. By choosing different values for ***j***, ***k***, ***v***, and ***w*** one can, of course, have rotationally identical or mirror image odd versions, but for even ***n*** there are two distinct versions.



One alternative is to change two equations, ***v*** = ***j*** and ***w*** = ***n***-2. This maintains the two sides of the largest triangle but alters the side that is used as the longest side from vertex ***n***/2 to ***n***/2-1 which is used to create the angle spanning two vertices (from 0-***n***/2-2 to 0-(***n***/2-1)-(***n***-2). The results for ***n*** = 8 and 10 are shown to the right, just beneath their counterparts above them. (For odd ***n*** (not shown), this adjusted ***v*** and ***w*** produces mirror images.)

Focus on the angle at 2. For ***n*** = 8 this angle now spans 2 vertices (not ***n***-3) and for ***n*** = 10 this angle now spans ***n***-3 vertices (not 2). In each instance, the angle at vertex 2+***n***/2 matches the angle at 2 for even ***n***. But note that for odd ***n***, the angle at 5 = 2+(***n***-1)/2 for ***n*** = 7 and at 7 = 2+(***n***+1)/2 for ***n*** = 9 is not the same as at vertex 2 in each image.

Counting Sharpest Most Obtuse Scalene Triangles. Since the triangles are sharpest most obtuse scalene triangles, each angle spanning a single vertex is located at one of the polygonal vertices and the other two angles look larger (they span 2 and ***n***-3 vertices) and one can readily recognize which is which. Note, for example, that at vertex 0 in all six images, the three lines at 0 create two angles, the one on the right spans a single vertex and the one on the left spans 2 for the first four, and the reverse is true for the last two. Given both constructions, vertex 2 has no sharpest angle. Were we to increase ***n*** to 11, then this would happen at vertex 3, and for ***n*** = 15 it would turn into vertex 4 where the sharpest angle count is 0. For even ***n***, this is true for both versions. In each case, for any ***n***, the *zig-zag* pattern starting at the vertex having 0 sharpest angles is 0, 1, 2, 2, …, 2, 1, 0. The only question is: *How many 2s are in the … middle?*

For ***n*** = 7, 8, 9, 10 above, visual inspection confirms that there are 3, 4, 5, and 6 2s in the middle, surrounded by counts of 1 and 0 on each side. Add 1 to ***n*** and that adds an additional vertex which increases the number of 2s in the middle by 1. Since it takes 2 vertices at each side to get to the first count of 2, there are ***n***-4 vertices with counts of 2. Therefore, the total count of triangles for sharpest scalene triangles is T(***n***) = 2 + 2(***n***-4) where the first 2 is the sum of the single counts on both sides (at vertices 1 and 6 for the left image and 3 and 8 for the right). Regrouping we see that T(***n***) = 2(***n***-3), which you can confirm by looking at the total count of triangles across images is 8, 10, 12, 14 for ***n*** = 7 through 10.