Difference Between Squares

One of the formulas that appears and reappears in various contexts is a very simple piece of algebra called the difference between squares formula.

$$\boldsymbol{a}^2 - \boldsymbol{b}^2 = (\boldsymbol{a}-\boldsymbol{b})(\boldsymbol{a}+\boldsymbol{b}).$$

In addition to being used in **ESA**, for example in sections E 8.1 and E8.5.2, we saw an elementary use of it to help students see patterns in multiplication tables in E21.3.

It is also used in counting exercises and elsewhere within **PwP**. Here and here are places within spirals where it is used. The second one has a bit of a primer on difference between squares.

It is also seen in counting triangles. Here is one interpretation of difference between squares and here is another.

This is an interesting use of the difference between squares formula to show something quite a bit more exotic. This is based on Chakerian and Erfle, *Up the Hill and Down Again*, equations 6 and 7 p. 255-6:

The Hill Formula is: $\mathbf{H}_{\boldsymbol{k}} = 1 + 2 + \ldots + (\boldsymbol{k}-1) + \boldsymbol{k} + (\boldsymbol{k}-1) + \ldots + 2 + 1 = \boldsymbol{k}^2$

We have also examined triangular numbers:

The $\boldsymbol{k}^{th}$ *triangular number* is: $\Delta_{\boldsymbol{k}} = 1 + 2 + \ldots + (\boldsymbol{k}-1) + \boldsymbol{k} = \boldsymbol{k}\cdot(\boldsymbol{k}+1)/2.$

$\mathbf{H}_{\boldsymbol{k}} = \Delta_{\boldsymbol{k}} + \Delta_{k-1}$ by construction. Further, $\Delta_{\boldsymbol{k}} - \Delta_{\boldsymbol{k}-1} = \boldsymbol{k}$.

Therefore, $\Delta_{\boldsymbol{k}}^2 - \Delta_{\boldsymbol{k}-1}^2 = (\Delta_{\boldsymbol{k}} + \Delta_{\boldsymbol{k}-1})\cdot(\Delta_{\boldsymbol{k}} - \Delta_{\boldsymbol{k}-1}) = \boldsymbol{k}^2\cdot\boldsymbol{k} = \boldsymbol{k}^3$.

If we sum both sides for $\boldsymbol{k} = 1, 2, \ldots, \boldsymbol{n}$ we obtain the sum of the first $\boldsymbol{n}$ cubes on one side and, because of telescoping cancellations, $\Delta_{\boldsymbol{n}}^2$ on the other side. This produces the result:

$$1^3 + 2^3 + \ldots + \boldsymbol{n}^3 = (1 + 2 + \ldots + \boldsymbol{n})^2.$$

In words, *the sum of the cubes is the square of the sum of the first* ***n*** *numbers.* This almost surreal result has been known for at least 1500 years! We see a geometric interpretation of this result in the counting cubes portion of the squares and cubes chapter.