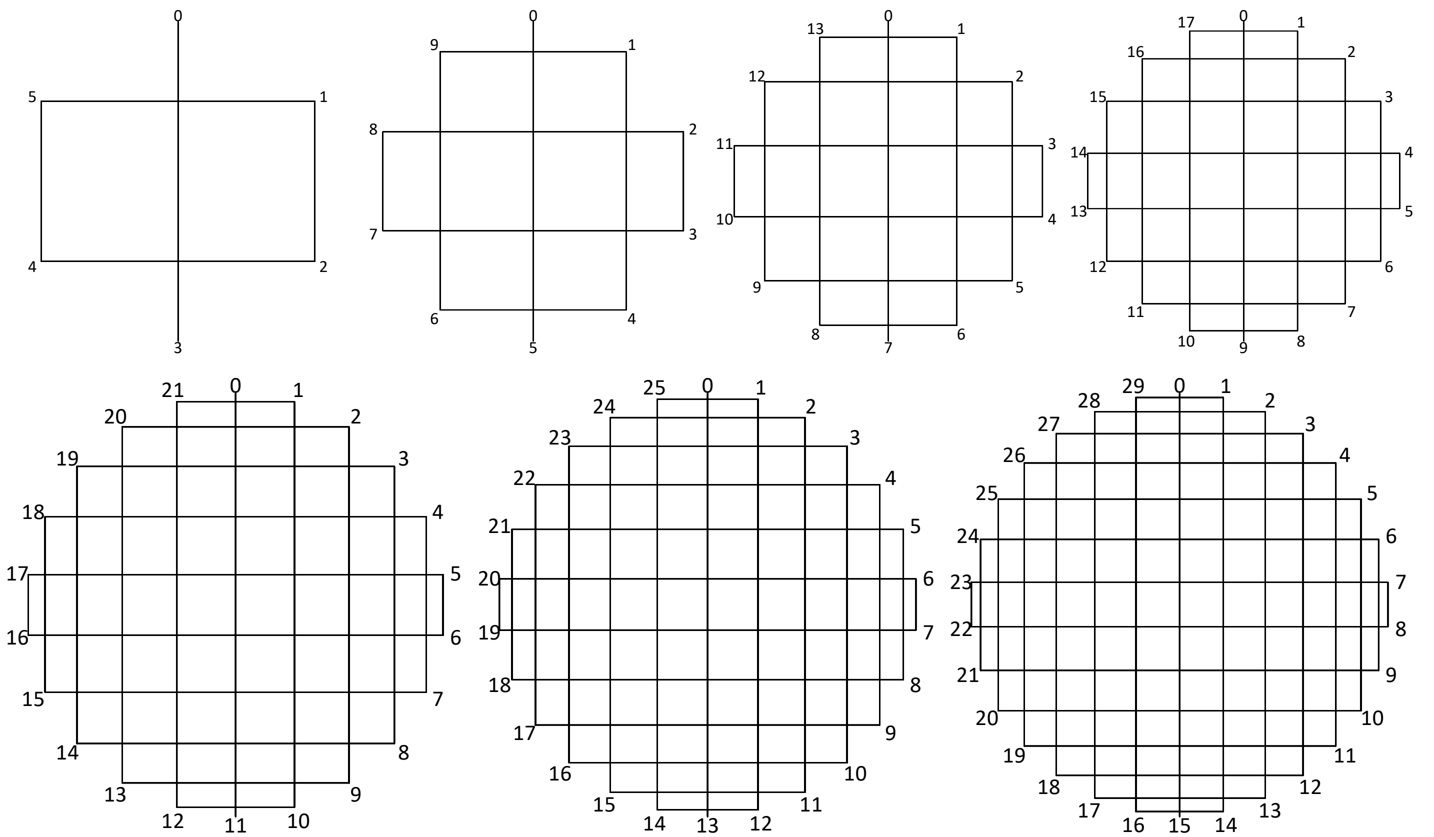
An Introduction to Rectangles

Rectangles require right angles and right angles require even ***n***. One way to proceed is to examine every second ***n*** and think of ***n*** = 2***k***+2 so that the first rectangles image since ***n*** = 4 is the first polygon with right angles. It also makes sense to start with horizonal and vertical lines in this situation. In this setting, odd ***k*** and even ***k*** rectangles images look different from one another. Therefore, it is better to analyze ***n*** = 4***k*** and ***n*** = 4***k***+2 images separately.

n* = 4*k*+2**. Note that here, ***n = 2·(2***k***+1) so ***n*** an odd multiple of 2. Rows 1 and 2 show the first 7 such images.



These images are similar to one another because each has a single diameter with 1 line (the vertical line from 0 to 2***k***+1), each has top and bottom horizontal lines that span two vertices (at 1 to ***n***-1 and 2***k*** to 2***k***+2) and vertices at the sides that span a single vertex (at ***k*** to ***k***+1 and 2***k***+1 to 3***k***+2). Any rectangular image on a 4***k***+2 polygon will be the same up to a rotation. (The same is not true for 4***k*** polygons where there are two distinct versions.)

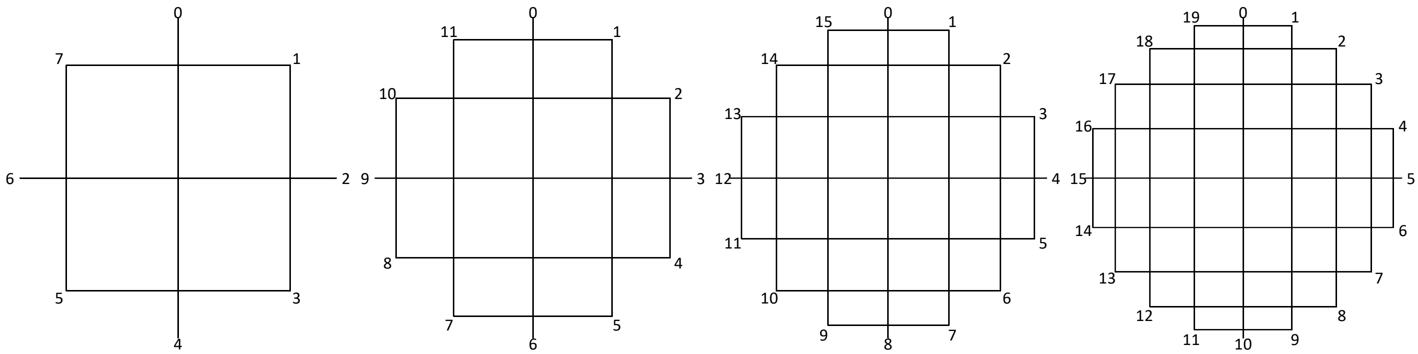
Counting Smallest 4*k*+2 Rectangles. Counting from either the top to bottom or side to side we see that the number of rectangles increases by two per row or column. If we count by row, we start with 2 but if we start by column, we start with 1. Both ways lead to the same result, and it is instructive to see both. The row 2 left, ***k*** = 5, ***n*** = 22 image has counts:

Top to Bottom: 2+4+6+8+10+8+6+4+2 = 2·(1+2+3+4+5+4+3+2+1) = $2·5^2$ = 50. In general, T(4***k***+2) = $2\boldsymbol{k}^2$ by the Hill Formula.

Side to Side: 1+3+5+7+9+9+7+5+3+1 = 2·(1+3+5+7+9) = $2·5^2$ = 50. In general, T(4***k***+2) = $2\boldsymbol{k}^2$ using gnomons.

n* = 4*k. The vertical and horizonal lines at 0-2***k*** and ***k***-3***k*** have no other lines at these vertices and the smallest rectangle counts are 2 at top, bottom, left and right. A second, slanted version of rectangles has no such vertices; all vertices have a right angle, and the smallest rectangle count is 1 at each of the four "corners" (0-1, ***k*** to ***k***+1, 2***k*** to 2***k***+1, and 3***k*** to 3***k***+1). Six images of each type are shown in rows 3-6. A quick comparison of ***n*** = 8, 3rd and 5th row left, shows that rectangle count totals differ for these two versions of images. If ***k*** = 1 images were shown, the horizontal has no rectangles, and the slanted one has 1 square.

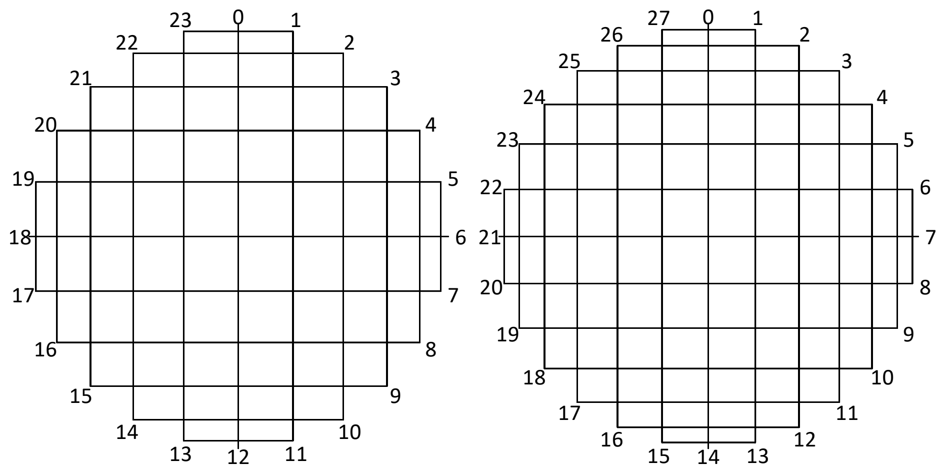
Counting Smallest 4*k* Rectangles. Both versions have the same pattern whether you count by rows or columns (although the rows and columns are, of course, slanted with the slanted version). As a result, visualize the counting whichever way you want. Both example images use ***n*** = 20, ***k*** = 5, the right image in rows 3 and 5 before asserting the general formula.

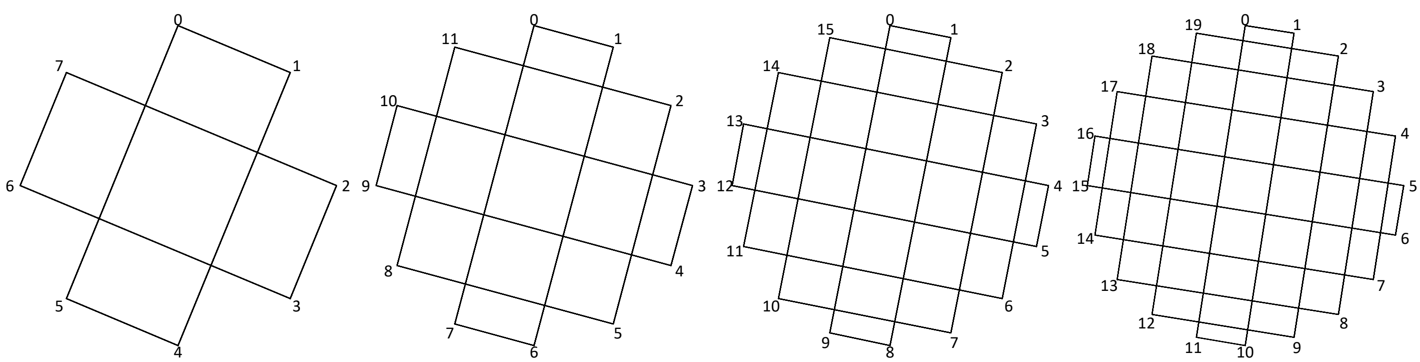


Total smallest horizontal rectangles image counts, $T_{\underline{H}}(4\boldsymbol{k})$ are:

$$
\begin{aligned}
T_{\underline{H}}(4\boldsymbol{k};\ \boldsymbol{k}{=}5) &= 2+4+6+8+8+6+4+2 \\
&= 2\cdot(1+2+3+4+4+3+2+1) \\
&= 4\cdot(1+2+3+4) \\
&= 4\Delta_4 = 4\cdot 4\cdot 5/2 = 40.
\end{aligned}
$$

In General:
$$
\begin{aligned}
T_{\underline{H}}(4\boldsymbol{k}) &= 4\Delta_{\boldsymbol{k}-1} \\
&= 4(\boldsymbol{k}-1)\boldsymbol{k}/2 \\
&= 2(\boldsymbol{k}-1)\boldsymbol{k}
\end{aligned}
$$



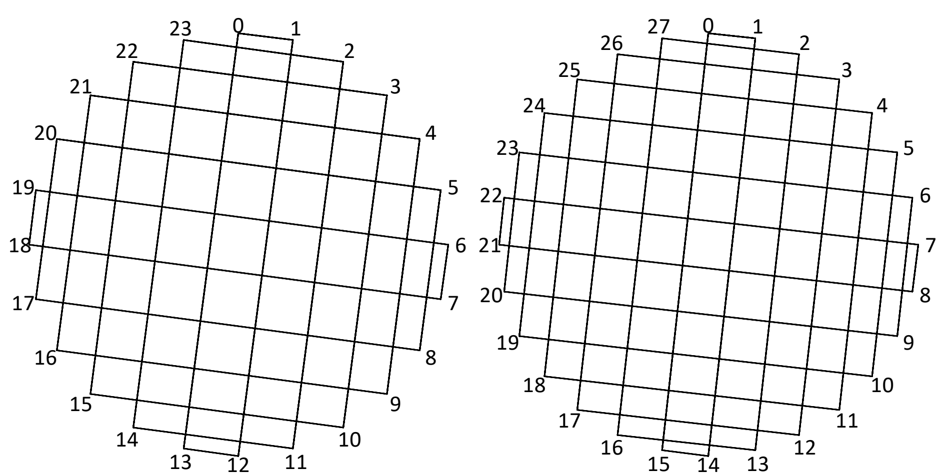


Total smallest slanted rectangles image counts, $T_S(4\boldsymbol{k})$ are:

$$
\begin{aligned}
T_S(4\boldsymbol{k};\ \boldsymbol{k}{=}5) &= 1+3+5+7+9+7+5+3+1 \\
&= 5^2 + 4^2 = 41 \text{ by gnomons.}
\end{aligned}
$$

In General:
$$
\begin{aligned}
T_S(4\boldsymbol{k}) &= \boldsymbol{k}^2 + (\boldsymbol{k}-1)^2 \\
&= \boldsymbol{k}^2 + \boldsymbol{k}^2 - 2\boldsymbol{k} + 1 \\
&= 2(\boldsymbol{k}-1)\boldsymbol{k} + 1.
\end{aligned}
$$

So, we see that the slanted $\boldsymbol{n} = 4\boldsymbol{k}$ images have one more rectangle than their horizontal counterparts. The 4 versus 5 or 0 versus 1 that we saw when $\boldsymbol{n}$ = 8 or 4 is true for all $\boldsymbol{k}$.



Morphing Images. One final point to note in moving from $\boldsymbol{n} = 4\boldsymbol{k}$ to $\boldsymbol{n} = 4\boldsymbol{k}+2$ is a gain of $2\boldsymbol{k}$ rectangles if we focus on horizonal images. Start with the $\boldsymbol{k}$ = 5, $\boldsymbol{n}$ = 20 right most image in row 3. Imagine each top half vertex moving up a bit, and bottom half moving down a bit to make room for 5 and 15 to split apart into a rectangle creating two more vertices. The result would be the $\boldsymbol{n}$ = 22, row 2 left image with the “new” rectangle 5-6-16-17 adding $2\boldsymbol{k}$ = 10 smallest rectangles. Continue adjusting vertices and add a line between 5-6 and 16-17 in $\boldsymbol{n}$ = 22 to obtain the $\boldsymbol{n}$ = 24, row 4 left image.