# The Sum of Squares Formula

The sum of squares of the first $\boldsymbol{k}$ numbers: $1^2 + 2^2 + \ldots + \boldsymbol{k}^2 = \boldsymbol{k}\cdot(\boldsymbol{k}+1)\cdot(2\boldsymbol{k}+1)/6$ .

One can prove this formula by induction but here is a geometric interpretation of this algebraic result based on the discussion surrounding Equation (19) p. 259 of Chakerian and Erfle, "*Up the Hill and Down Again,*" **College Mathematics Journal**, Vol. 64, No. 4, September 2023.

Since this book is more casual in nature than a journal article, the argument is built out a bit more here. In particular, the argument is presented for two specific values of $\boldsymbol{n}$ before asserting a general answer.

Consider a triangle of numbers with each row being a series of odd numbers. The first row is the first odd, the second the first two and so on. The $\boldsymbol{k}^{th}$ row would have the first $\boldsymbol{k}$ odd numbers, the last of which is $2\boldsymbol{k}-1$. Two such triangles are shown below, the first with three rows, and the second with four.

We know from gnomons that the first $\boldsymbol{k}$ odd numbers is $\boldsymbol{k}^2$ so the sum of numbers in a $\boldsymbol{k}$ row triangle is $1^2+2^2+\ldots+\boldsymbol{k}^2$.

Let **S** be this sum. If we write **S** three ways as noted in the first example, vertically aligned with one another, we end up with the same kind of magic that Gauss saw in summing the first $\boldsymbol{n}$ numbers; these three ways create columns that sum to the same number.

The first and last columns show us that this number is the bottom right number from the triangle plus 2 (5+2 with 3 rows, 7+2 with 4 rows, and $2\boldsymbol{k}-1+2$ with $\boldsymbol{k}$ rows). Therefore, in the general setting the number is $2\boldsymbol{k}+1$.

We can pull out the common 7, 9, or $2\boldsymbol{k}+1$ from each term and we are left with a lot of 1s. Each row (of the triangle) has a different number of 1s (noted in the parentheses in the first example) with the $\boldsymbol{k}^{th}$ row having $\boldsymbol{k}$ 1s. Rewriting these 1s, we have $1+2+3+\ldots+\boldsymbol{k}$ or the $\boldsymbol{k}^{th}$ triangular number, $\Delta_{\boldsymbol{k}}$, of 1s.

$$\begin{array}{ccccc|l} & & 1 & & & = 1^2 \\ & 1 & & 3 & & = 2^2 \\ 1 & & 3 & & 5 & = 3^2 \\ \hline & & & & & \Sigma = \mathbf{S} \end{array}$$

$$\begin{array}{ll} \mathbf{S} = 1 + (\ 1 + 3\ ) + (\ 1 + 3 + 5\ ) & \text{by row left to right} \\ \mathbf{S} = 1 + (\ 3 + 1\ ) + (\ 5 + 3 + 1\ ) & \text{by row right to left} \\ \mathbf{S} = 5 + (\ 3 + 3\ ) + (\ 1 + 1 + 1\ ) & \text{by diagonal} \\ \hline 3\mathbf{S} = 7 + (\ 7 + 7\ ) + (\ 7 + 7 + 7\ ) & = 7 + 7(1+1) + 7(1+1+1) = 7(1+2+3) = 7\Delta_3 \end{array}$$

So $3\mathbf{S} = 7\cdot 6 = 42$, $\mathbf{S} = 14$

$$\begin{array}{ccccccc|l} & & & 1 & & & & = 1^2 \\ & & 1 & & 3 & & & = 2^2 \\ & 1 & & 3 & & 5 & & = 3^2 \\ 1 & & 3 & & 5 & & 7 & = 4^2 \\ \hline & & & & & & & \Sigma = \mathbf{S} \end{array}$$

$$\begin{array}{l} \mathbf{S} = 1 + (\ 1 + 3\ ) + (\ 1 + 3 + 5\ ) + (\ 1 + 3 + 5 + 7\ ) \\ \mathbf{S} = 1 + (\ 3 + 1\ ) + (\ 5 + 3 + 1\ ) + (\ 7 + 5 + 3 + 1\ ) \\ \mathbf{S} = 7 + (\ 5 + 5\ ) + (\ 3 + 3 + 3\ ) + (\ 1 + 1 + 1 + 1\ ) \\ \hline 3\mathbf{S} = 9 + (\ 9 + 9\ ) + (\ 9 + 9 + 9\ ) + (\ 9 + 9 + 9 + 9\ ) = 9\Delta_4 \end{array}$$

$\mathbf{S} = 9\cdot 10/3 = 30$

Putting this all together we have:

$$3\mathbf{S} = (2\boldsymbol{k}+1)\Delta_{\boldsymbol{k}} = (2\boldsymbol{k}+1)\boldsymbol{k}(\boldsymbol{k}+1)/2.$$

Dividing by 3 and recalling that **S** is the sum of the first $\boldsymbol{k}$ squares we have:

$$1^2 + 2^2 + \ldots + \boldsymbol{k}^2 = \boldsymbol{k}\cdot(\boldsymbol{k}+1)\cdot(2\boldsymbol{k}+1)/6.$$