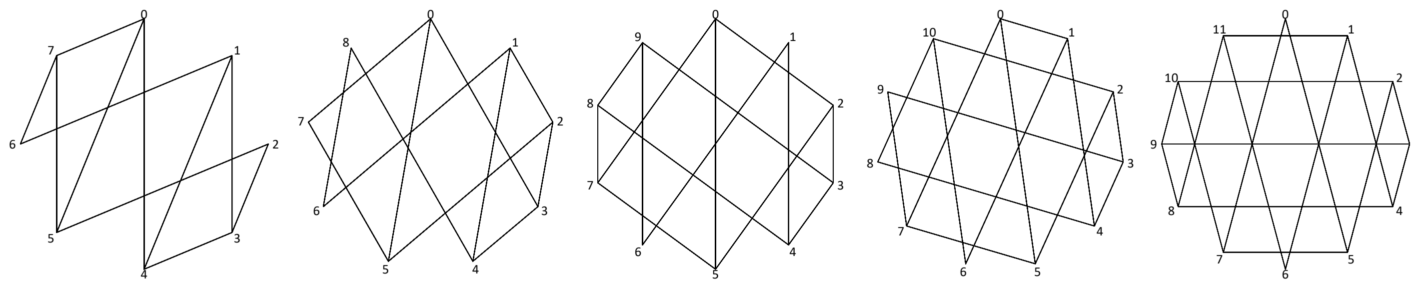
Analyzing a Fixed Set of Parameters as *n* Changes, 0-5-7 Images

This is a very open-ended and simple series of images to work on because the idea is to simply put in numbers for ***JKVW*** rather than have them tied to ***n***. Then consider what happens as ***n*** increases. All it takes is proposing a set of numbers.

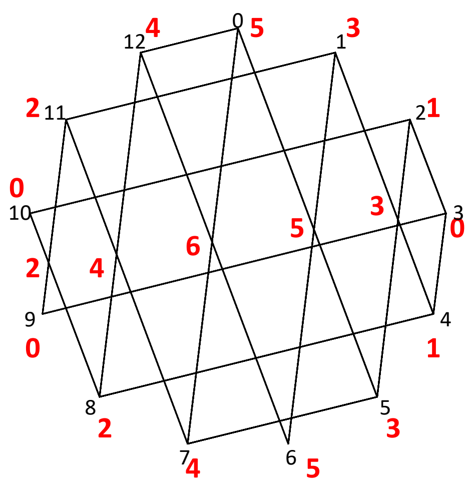
Let ***j*** = 5, ***k*** = 7, ***v*** = 5 and ***w*** = 7. Since ***v***-***w*** is tied to ***j*** and ***k***, the image always includes triangle 0-***j***-***k***, here 0-5-7 if ***n*** ≥ 8.

The first row shows ***n*** = 8 to 12. By construction, the 5-0-7 angle spans 2 vertices and the 0-7-5 angle spans 5 vertices. As ***n*** increases, the 0-5-7 angle expands, and it always spans ***n***-7 vertices, from 1 to 5 in these first five images.

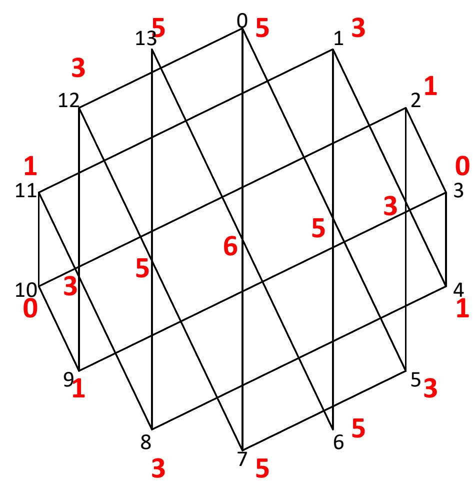


For ***n*** > 12, the middle-sized angle remains fixed at 5, regardless of ***n***. We learned at the end of the last chapter that, by focusing on the middle-sized angle, we were able to quickly analyze any sharpest angle image. We have set-up the next easiest situation; we want to examine the second-sharpest images. The next 4 images show the patterns that develop.

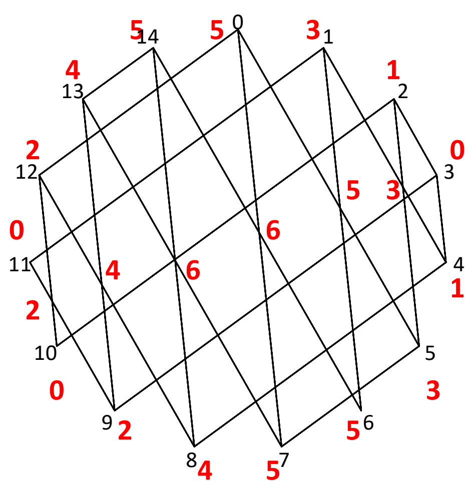
n = 13 50



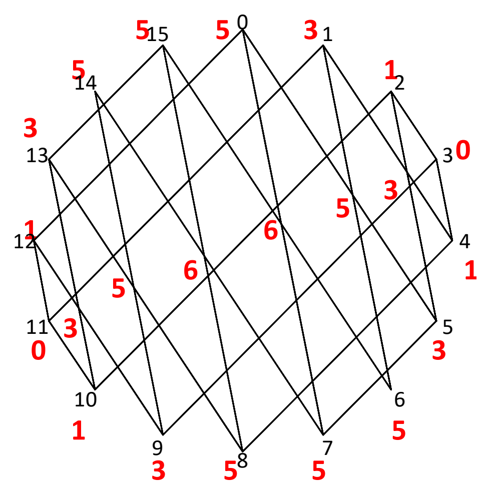
n = 14 58



n = 15 66



n = 16 74



Clearly, evens are different from odds.

Odd *n*. At top and bottom half from side to side around the outside you see 1+2+3+4+5 = Δ_5 for ***n*** = 13, and for ***n*** = 15 we have Δ_5+5.

The middle has 2+3+4+5+6 = Δ_6-1 for 13 and 6 more than that for 15.

If we check n = 17 you will see that the only difference is that there is one more 5 at top and bottom and one more 6 in the middle. Therefore, if we let ***n*** = 2***k***+1 then 13 is ***k*** = 6.

$T_{odd}(n = 2k+1) = 2\Delta_5+\Delta_6-1+16(k-6)$
$= 2(5\cdot 6)/2+(6\cdot 7)/2-1+16(k-6)$
$= 50+16(k-6)$ for $k > 5$.

Even *n*. The outside and inside patterns are different but the expansion is the same. For even ***n*** = 14 there are 4 (1+3+5)s or 36 total around the outside and 3+5+6+5+3 or 22 on the inside for a total of 58. And as ***n*** increases by 2, 16 triangles are added to the middle (notice for ***n*** = 16 that there are 3 5s at top and bottom and 2 6s inside). So,

$T_{even}(n = 2k+2) = 58+16(k-6)$ for $k > 5$.

Both equations require images where the two smallest angles span 2 and 5 vertices.

An aside on concurrence. One final question you might have is whether there is really 1 less triangle in the ***n*** = 15 image because it looks like the lines 2-10, 14-8 and 13-7 all intersect at a single point. This is called a *concurrence*. In fact, that is not the case, and it is easy to show that this is true using a bit of trigonometry and the equation of a line. A section of Mathematical Tidbits discusses an *Excel* file that easily accomplishes this task.

The coordinates of the 2-10 and 14-8 lines intersection is:	(-0.3013, -0.0897)
The coordinates of the 2-10 and 13-7 lines intersection is:	(-0.3039, -0.0916)
The coordinates of the 13-7 and 14-8 lines intersection is:	(-0.3005, -0.0976)

So, yes, it is a triangle, and yes, it is small. By looking at these values we see that the line 2-10 is above the apex intersection of 13-7 and 14-8 (since this y value is the smallest of the three intersection y values) so that the small triangle is pointed down. There are 2 triangles below the apex and 4 above or **6** triangles using this apex.

We did not focus on the ***n*** = 8 to 12 images here. The first four had middle angle spanning from 2 to 4 vertices and the last one, ***n*** = 12 is special in its own way because it produces isosceles rather than scalene triangles. It is an example of a second-sharpest isosceles triangle on even ***n*** and it will be examined elsewhere within this chapter. In that image, every interior apex is a point of concurrence since each interior apex is on the diameter from 3 to 9.