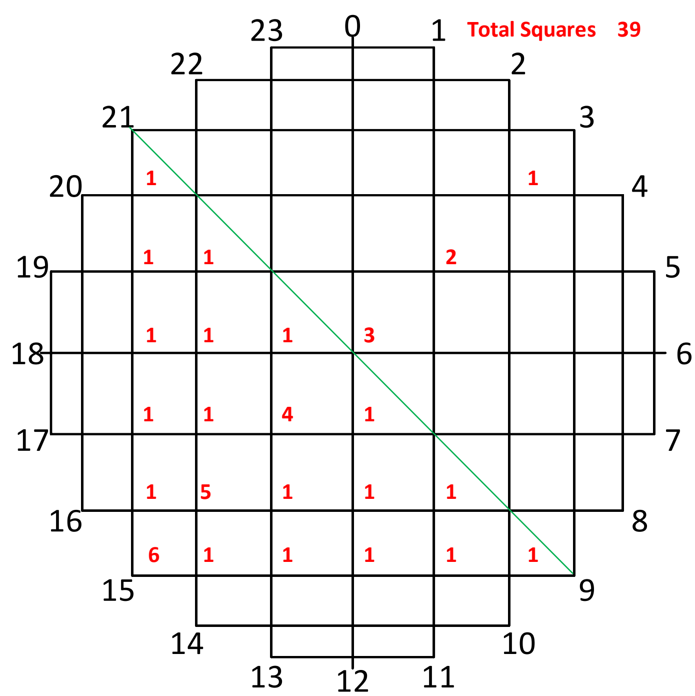
Counting Squares in Horizontal Rectangles Images

Although the question of counting all rectangles of various sizes in rectangles images appears to be quite complex, the question of counting squares in rectangles images is much less of a problem. From Squares in Rectangles we know:

1. There are no squares if n is an odd multiple of 2.
2. The number of vertices spanned by the largest square in a horizontal rectangles image is always an even number and it increases by two for every increase in n of 8. The first n for which there is a horizontal square is $n = 8$.
3. When n is an even multiple of 4, $n = 8k$, the largest rectangle is a $2k$ square connecting vertices k-$3k$-$5k$-$7k$ of the n-gon (note that each corner is separated from the previous corner by $2k$ vertices, see 3-9-15-21 below).
4. When n is an odd multiple of 4, $n = 8k+4$, the largest rectangle is an internal $2k$ square whose upper right corner is on the diameter midway between k and $k+1$ (3 and 4 given $k = 3$, $n = 28$) and whose upper left corner is on the diameter midway between $7k$ and $7k+1$ (24 and 25 and $n = 28$, row 4 right image in Introduction to Rectangles).

This is a comparatively easy problem to answer because we know where to look to find squares. Any square must have at least one set of opposing corners on a line that is a diameter of the circle to ensure that height and width are equal. These are the **green lines of symmetry** shown in Squares with Rectangles.

Counting Squares. 3 and 4 above suggests the same formula applies for even and odd multiples of 4. Let $S_H(k)$ be the number of squares of various sizes in horizontal rectangles images given $n = 8k$ and $n = 8k+4$. Let the lower left corner of each rectangle be the distinguished point used for counting squares. The image shows squares counts for $n = 24$, $k = 3$.



The squares counts on the diameter from 3 to 15 are the numbers from **1** to **6** and represent all squares with opposing corners on this diagonal. There are $21 = 6\cdot 7/2$ squares with lower left corner on this diameter using the triangular numbers formula.

The **1**s in cells beneath **the green diameter from 9 to 21** represent the lower left corner of squares that are not on the diameter from 3 to 15 but have upper left and lower right corners on the **9-21 diameter**. There are two triangles of **1**s, and both have the same pattern of 1, 3, 5, or the first 3 odd numbers. We know from gnomons that this sum is 3^2 or 9 so we have 18 total **1**s beneath the **green diameter**.

In total there are $S_H(3) = 21 + 18 = 39$ squares in the image.

A General Rule. When $k = 2$ so $n = 16$ and 20 we have a 4 by 4 square with numbers from **1** to **4** on the diagonal with upper right corner at 2 or between 2 and 3. Additionally, there is one less row of odd **1**s (or 1 and 3). The reverse happens when k increases. The numbers on the upward sloping diagonal starting at or near k increase by 2 and a new row with $2k-1$ **1**s is added to the left side and bottom side of the largest square. For general k, there are $\triangle_{2k}$ squares with lower left corner on the upward sloping diagonal starting near k, and an additional $2k^2$ triangles of **1**s that represent squares with lower left corner off that diagonal but whose upper left corner is on the downward sloping diagonal starting near vertex $7k$. In net we see that for general k, there are a total number of squares of various sizes in the horizontal rectangles images of the form $n = 8k$ and $n = 8k+4$ of:

$$S_H(k) = \triangle_{2k} + 2k^2 = 2k(2k+1)/2 + 2k^2 = 2k^2 + k + 2k^2 = 4k^2 + k = k(4k+1).$$

Centrally located squares. All but k of these squares have opposing corners on one but not both of the largest square's diagonals. Put another way, k of these squares are *centrally located*, and they have dimensions 2×2, 4×4, …, $2k\times 2k$.