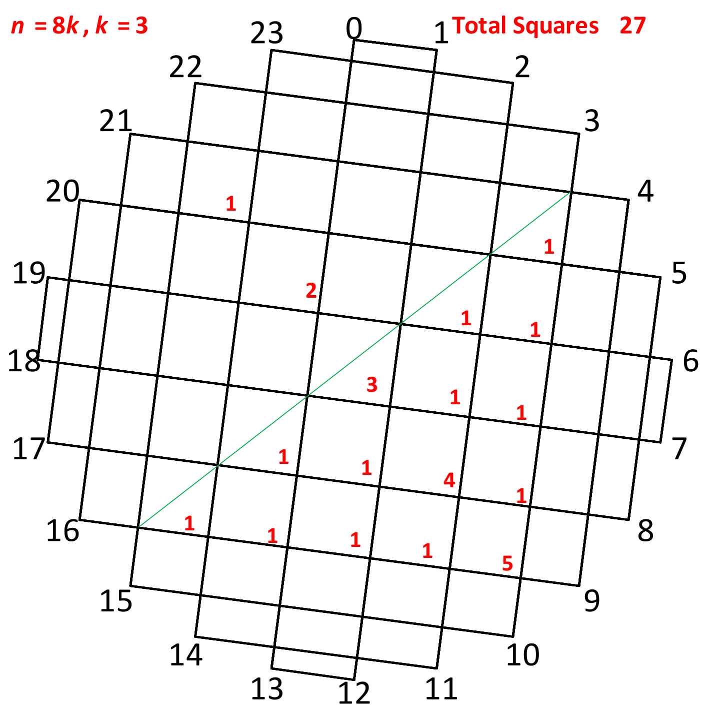
Counting Squares in *Slanted* Rectangles Images

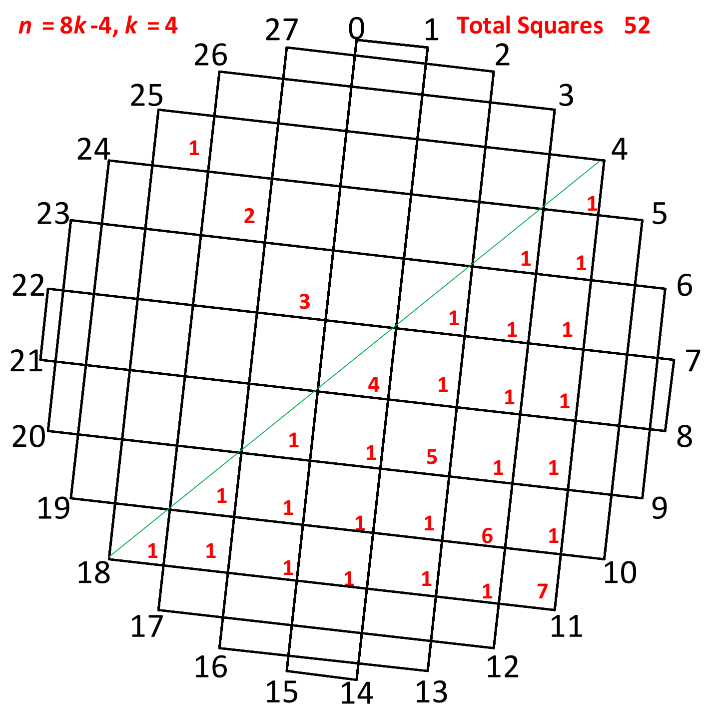
We noted at the end of Squares in Rectangles Images that slanted images produce similar, but odd, results. By examining the images in rows 5 and 6 of Introduction to Rectangles we readily see that squares in slanted images are governed by similar rules. As with horizontal images, there are no squares if n is an odd multiple of 2. However,

1. The number of vertices spanned by the largest square in a slanted rectangles image is always an odd number and it increases by two for every increase of 8 in n. The first n for which there is a slanted square is $n = 4$.
2. When n is an odd multiple of 4, $n = 8k$-4, the largest rectangle is a 2k-1 square whose upper right corner is at k. The vertices of this square are k - 3k-1 - 5k-2 - 7k-3 of the n-gon (note that each corner is separated from the previous corner by 2k-1 vertices). In the right image the 7×7 square has vertices 4-11-18-25 given $k = 4$.
3. When n is an even multiple of 4, $n = 8k$, the upper right corner is on the diameter midway between vertices k and k+1 (between 3 and 4 given $k = 3$ and $n = 24$ in the left image) and whose upper left corner is on the diameter midway between 7k and 7k+1 (21 and 22 in that image).

Like counting horizontal squares, this problem is easy to answer because we know where to look to find squares. Any square must have at least one set of opposing corners on a line that is a diameter of the circle to ensure that height and width are equal. These are lines of symmetry just like the **green lines of symmetry** shown in Squares with Rectangles.

Counting Squares. 2 and 3 above suggests the same formula applies for even and odd multiples of 4. Let $S_S(k)$ be the number of squares of various sizes in *slanted* rectangles images given $n = 8k$ and $n = 8k$-4. Let the bottom corner (lower right) of each rectangle be the distinguished point used for counting squares. Squares counts are in **red**.





The squares counts on the downward sloping diagonals are the numbers from **1** to **5** or 15 in the left image and **1** to **7** or 28 in the right image given $k = 3$ and $k = 4$, respectively. For general k, these are the numbers from 1 to 2k-1. The sum of these numbers is $\triangle_{2k-1} = (2k-1)2k/2 = 2k^2-k$.

The **1**s in cells beneath **the upward sloping green diagonals** represent the bottom corner of squares that have upper right and lower left corners on the **green diagonal**. There are two triangles of ones, with an increasing number of even **1**s in k-1 rows (2, 4 = 2 times (1, 2), or a total of 12 in the left image and 2, 4, 6 = 2 times (1, 2, 3), or a total of 24 in the right image). In general, each image has $4\triangle_{k-1} = 4(k-1)k/2 = 2k^2-2k$ squares in two triangles of **1**s. Adding these two pieces together, we see that the total number of squares in a slanted image given $n = 8k$ and $n = 8k$-4 is:

$S_S(k) = \triangle_{2k-1} + 4\triangle_{k-1} = 2k^2-k + 2k^2-2k = 4k^2-3k = k(4k-3)$, k of which are centrally located with side lengths 1, 3, …, 2k-1.

NOTE. These squares counts are quite different from those examined elsewhere with sides subdivided into k equal parts.