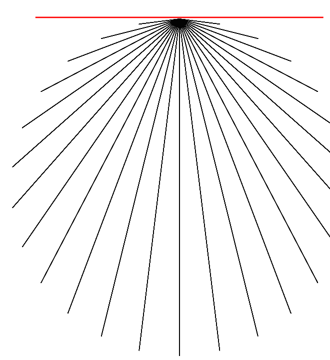
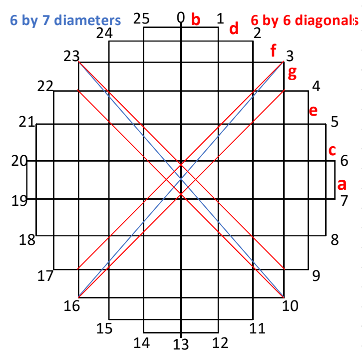
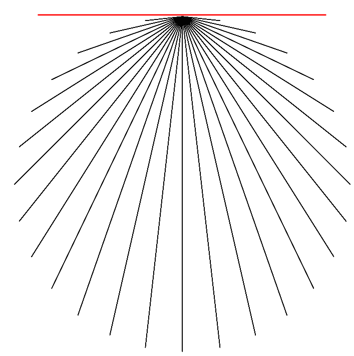
MA. An Angular Approach to Squares in Rectangles

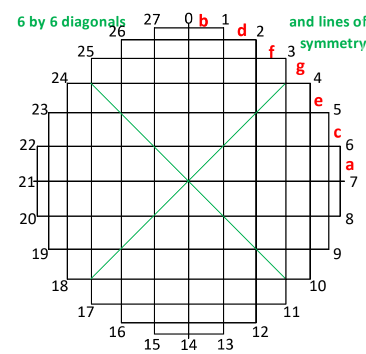
In the previous section we argued that there are no squares in ***n*** = 4***k***+2 rectangles images because side lengths will never be equal, but we do find squares in ***n*** = 4***k*** rectangles images. The reason for this distinction involves how the angle of a right triangle changes for these two types of rectangles images. As the angle increases from 0, the opposing side of the right triangle increases and the adjacent side decreases. Initially, the opposing side will be very small relative to the adjacent side, but once the angle passes 45° the two opposing side is larger than the adjacent side. At 45°, x = y = $2^{0.5}/2$ = 0.707107… . Our interest is in the size of the horizontal and vertical sides derived from these angular changes.

We are going to explore this distinction using an idea from the Cardioids chapter: we can consider vertices as directions. This is another method to visualize vertices in addition to the standard way of visualizing vertices radially around the center of the circle. The first and third images explore *vertices as directions*, **VaD**, for ***n*** = 26 and ***n*** = 28. The second and fourth are the images from the previous section for these values of ***n***, ***n*** = 4***k***+2 with ***k*** = 6 on the left and ***n*** = 4***k*** with ***k*** = 7 on the right. Both are examined in the table based on x and y values and **red lettered lengths** from those images.









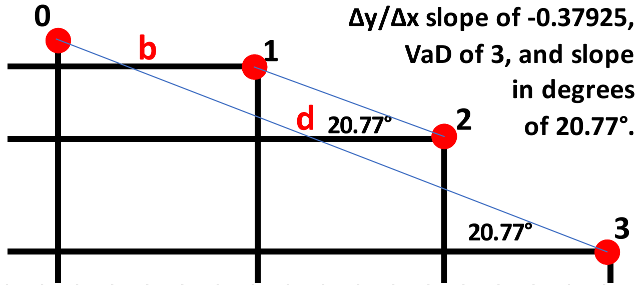
Vertices as Directions. If we consider the lines created by connecting 0 to the other ***n***-1 vertices of an ***n***-gon we have mapped out ***n***-1 directions relative to 0. The final direction is a horizontal line at 0 in **red**. The result is a *circle fan* with ***n***-1 lines at regular angles (due to the inscribed angle theorem) of 180***j***/***n***° relative to the flat top in column 4 in degrees and as slope in column 7 (and shown in image 1 and 3) with VaD labels in column 6. A circle fan ends up a looking a bit like a 180° protractor with flat side at the top or a palm frond or a ribbed clam shell.

Col.	1	2	3	4	5	6	7	8	9	10	11	12	13
26-gon vertices in 1st quadrant*		x and y based on sine		Slope in degrees from		Slope as Rise over Run and							
		and cosine functions		Top°	Center°	Vertices as Directions, VaD				length^	1/slope		length^
	j	x	y	= 180*j*/26°	= 360*j*/26°	VaD	slope m = Δy/Δx				= Δx/Δy		
	0	0	1	0	0	**0**	From 0	From *j*-1	VaD		From *j*-1	VaD	
	1	0.2393157	0.9709418	6.9230769	13.846154	**1**	**-0.12142**	**-0.12142**	**1**	b	-8.23574		
	2	0.4647232	0.885456	13.846154	27.692308	**2**	**-0.24648**	**-0.37925**	**3**	d	-2.63678		
	3	0.6631227	0.7485107	20.769231	41.538462	**3**	**-0.37925**	**-0.69025**	**5**	f	-1.44875		
	4	0.8229839	0.5680647	27.692308	55.384615	**4**	**-0.52484**	**-1.12877**	**7**		**-0.88592**	**6**	g
	5	0.9350162	0.3546049	34.615385	69.230769	**5**	**-0.69025**	-1.90534			**-0.52484**	**4**	e
	6	0.9927089	0.1205367	41.538462	83.076923	**6**	**-0.88592**	-4.05716			**-0.24648**	**2**	c
	7	0.9927089	-0.120537	48.461538	96.923077	**7**	**-1.12877**	divide by 0			**0**	**0**	a
	*** 7th point is in 4th quadrant for *n* = 26. See two left images.**									^Based on 26-gon image.			
28-gon vertices in 1st quadrant~				Top°	Center°	Vertices as Directions, VaD				length^	1/slope		length^
	j	x	y	= 180*j*/28°	= 360*j*/28°	VaD	slope m = Δy/Δx				= Δx/Δy		
	0	0	1	0	0	**0**	From 0	From *j*-1	VaD		From *j*-1	VaD	
	1	0.2225209	0.9749279	6.4285714	12.857143	**1**	**-0.11267**	**-0.11267**	**1**	b	-8.87525		
	2	0.4338837	0.9009689	12.857143	25.714286	**2**	**-0.22824**	**-0.34992**	**3**	d	-2.85784		
	3	0.6234898	0.7818315	19.285714	38.571429	**3**	**-0.34992**	**-0.62834**	**5**	f	-1.59149		
	4	0.7818315	0.6234898	25.714286	51.428571	**4**	**-0.48157**	**-1**	**7**	g	**-1**	**7**	g
	5	0.9009689	0.4338837	32.142857	64.285714	**5**	**-0.62834**	-1.59149			**-0.62834**	**5**	e
	6	0.9749279	0.2225209	38.571429	77.142857	**6**	**-0.79747**	-2.85784			**-0.34992**	**3**	c
	7	1	0	45	90	**7**	**-1**	-8.87525			**-0.11267**	**1**	a
	~ 7 = *k* is on the x axis for *n* = 4*k*. See two right images									^Based on 28-gon image.			
° Slope in degrees from horizontal for Top (Circle Fan) and around circle for Center													
For ***n*** = 4***k***, these Center Slopes come in pairs (***j*** and ***k***-***j***) of complementary angles: 1+6 = 2+5 = 3+4 = 90°.													

The table examines attributes of the first quarter of each fan since these represent the location of the upper right corner of the largest rectangles in a horizontal rectangles image near vertex ***k***/2 (at vertex 3 for ***n*** = 26 and between 3 and 4 for ***n*** = 28) and they show all side lengths that comprise the horizontal and vertical pieces of the largest rectangle due to symmetry about the vertical and horizontal diameters.

The VaD directions provide a guide for directions for each side of an ***n***-gon. If we call the side from ***j***-1 to ***j*** the ***j***th side then the rise over run slope (Δy/Δx) of that side, calculated in column 8 is the VaD of vertex 2***j***-1 as seen in columns 6-10. The result seems surprising, but it is easy to understand. Because of the parallel lines nature of adding and subtracting vertices, we know that the line from ***j***-1 to ***j*** is parallel to the line connecting vertex 0 = ***j***-1 - (***j***-1) to ***j*** + (***j***-1) = 2***j***-1 (see green cells). This continues around the polygon with the only amendment being that if 2***j***-1 > ***n*** then VaD is MOD(2***j***-1,***n***) so that, for example, the side from ***n***/2 to ***n***/2+1 has VaD = 1, just like 0-1 since 2(***n***/2+1)-1 = ***n***+2-1 = 1 mod ***n***.

The blow-up at right of vertices 0-3 shows these three notions of slope (as rise over run, as VaD, and in degrees) based on information from the ***n*** = 26 portion of the table. We know the distance between each ***n***-gon vertex is the same. In this instance, that distance is y_6-y_7 = 0.2410734 = **a**, which is the same as the hypotenuse length between 1-2 while the adjacent leg at 2, based on an angle of 20.77° is **d** = 0.2254075 = x_2-x_1. This angle is the same as VaD 3 because **1-2** and **0-3** are parallel lines.



Why 1/slope? Once we consider ***j*** in the range ***k***/2 < ***j*** < ***k*** we move past the vertex whose upper right corner is the one with the largest rectangles. Vertex to vertex slope in column 8 becomes larger than 1 in magnitude. This is the region where we need to consider height rather than width because now, we are measuring the side of the largest rectangle rather than its top. We do this by examining 1/slope in column 11. Here is where the difference between odd multiples of 2, ***n*** = 4***k***+2, and even multiples of 2, ***n*** = 4***k***, becomes crystal clear.

The vertical lengths backwards from 7th to 6th to 5th to 4th are seen in the 26-gon figure second from the left and in the top half of the table as **a**, **c**, **e**, **g**. The first, **a**, is the distance between the 26-gon's vertices since vertices 6 and 7 are vertically aligned. The rest are legs of right triangles with **a** as the hypotenuse. The key here is that in terms of 1/slope the angles for **c**, **e**, and **g** are the 2nd, 4th, and 6th VaD slopes from columns 6 and 7, shown in shades of blue in columns 11 and 12. *There is no way in which the vertical longer legs of these triangles, could equal the horizontal longer legs of the sides in the top half of the first quadrant since even VaD slopes and odd VaD slopes will NEVER equal one another!*

By contrast, when ***n*** = 4***k*** like the ***n*** = 28 version examined in the bottom half of the table and shown in the right image, we see symmetry moving from 0 towards the middle (at ***k***/2) and from larger ***j*** backward from ***j*** = ***k*** towards the middle. This symmetry extends whenever ***n*** = 4***k***, not simply for ***k*** = 7 or ***n*** = 28. The reason for this symmetry is the 90° rotational symmetry discussed in the last section which leads to the conclusion that, for vertices ***j*** ≤ ***k*** in the first quadrant, $x_j = y_{k-j}$ and $y_j = x_{k-j}$. *Each of the horizontal sides in the top half is matched by a vertical side in the bottom half of this quadrant.*

Will a polygon ever use all VaD directions? One interesting pattern we noted above is that only odd VaD directions are used in describing slopes of a polygon. What happens to the even VaD slopes and why are only half the slopes used in the first place? Let's deal with the second question first.

Only half the slopes are being used because there is 180° rotational symmetry of an ***n***-gon given even ***n***. The bottom looks just like the top from a slope perspective; the first vertex after ***n***/2 has the same VaD slope as the first vertex as noted above. The same is true for all subsequent vertices on the ***j*** > ***n***/2 side of the ***n***-gon. Each has a parallel counterpart on the other side of an even ***n***-gon.

The situation is different for odd ***n***-gons. Odd ***n***-gons do not have 180° rotational symmetry and one of the first things we noticed about odd polygons was that they have a flat bottom. If ***n*** = 2***k***+1 the line from ***k*** to ***k***+1 is horizontal, or VaD = 0. The line from ***k***+1 to ***k***+2 has VaD slope 2, and so on; the last line of the ***n***-gon, from 2**k** to 2***k***+1 has VaD slope 2***k*** since 2(2***k***+1)-1 = (2***k***+1) + (2***k***+1)-1 = ***n*** + 2***k*** = 2***k*** MOD ***n***. So, we see that the right half of an odd ***n***-gon has odd VaD slopes and the left half of an odd ***n***-gon has even VaD slopes. All VaD slopes are seen in an odd ***n***-gon, including 0. Put another way, there are NO parallel counterpart sides on the other side of an odd ***n***-gon.