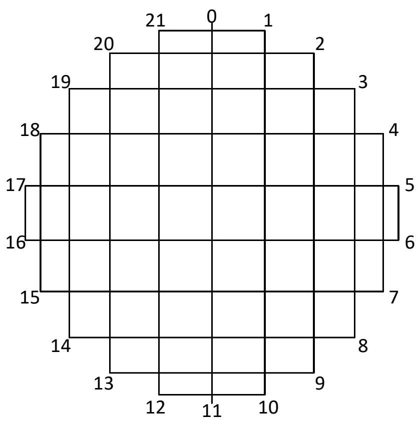
On Largest Rectangles

The rectangles images laid out in rows 1-6 of the last section can be analyzed in additional ways. The strategy laid out in earlier chapters of finding the total number of triangles of any size in an image appears to be a much more complex problem because of the varying nature of the size of rectangles available across image. We can see an example of this issue in About Distinguished Points which looks at ***n*** = 12. We proceed by examining other less complex questions here.

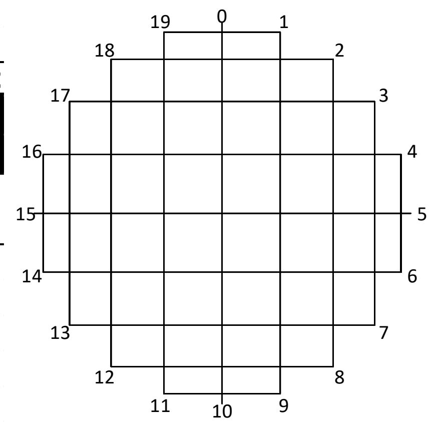
Defining the Largest Rectangle. Let the size of the rectangle, ***r***, be the number of vertices spanned in each direction multiplied times one another. Consider vertices in the first quadrant (vertices 1 to ***k*** given both ***n*** = 4***k***+2 and ***n*** = 4***k*** horizontal images in rows 1-4). Let ***j*** be such a vertex. Then the largest rectangle associated with that vertex is 2***j*** wide by ***n***/2-2***j*** high. Which vertex ***j*** produces the largest value of the product 2***j***(***n***/2-2***j***)? An *Excel* file helps with this analysis.

We begin with the ***k*** = 5, ***n*** = 22 and ***n*** = 20 exemplars from the previous section, reproduced here for ease of discussion. Included between the images is data from the *Excel* file which will be discussed in a more general context below. This file restricts attention to the right half of each image because each is symmetric about its vertical diameter. It focuses on the rectangles created using vertices in the first quadrant from 1 to ***k*** in each image.



Largest right half rectangles, ***r***/2, for even ***n***

Odd multiples of 2		**30**	**Even multiples of 2**		
n = 4***k*** +2 = 22		Max ***r***	***n*** = 4***k*** = 20		**24**
5 = ***k***	max ***j*** = 3	***r*** /2 =			Max ***r***
j, vertices	***h*** =2***k***+1-2***j***	***jh***		***h*** =	***r*** /2 =
1-***k*** (width)	(height)	(total/2)	***j***	2***k***-2***j***	***jh***
1	9	9	1	8	8
2	7	14	2	6	12
3	5	15	3	4	12
4	3	12	4	2	8
5	1	5			



n *= 4**k**+2.* The ***n*** = 22 image at left had 50 smallest rectangles, but we see 30 of those rectangles are contained in the rectangle ***r*** whose vertices are 3-8-14-19. Notice where the other 20 smallest rectangles come from in this image; 6 are above and 6 are below while 4 are to the right and 4 are to the left of the largest rectangle. The even/odd nature of counting the right half from the top and bottom versus side to side rectangle counts when ***n*** is an odd multiple of 2 means that all height counts are odd numbers. The product of height and width therefore alternates between even and odd because width (vertices ***j***) alternates between even and odd.

n *= 4**k**.* The ***n*** = 20 image at right had 40 smallest rectangles, but we see that 24 of them are contained in the largest rectangle ***r***. In this instance, there are two largest rectangles. One is 2-8-12-18, the other is 3-7-13-17. More generally, in every odd ***k*** situation the vertex producing the largest ***r*** is on either side of ***k***/2 at ***j*** = (***k***-1)/2 and ***j*** = (***k***+1)/2.

Had we shown an even ***k*** for the even multiples of 2, we would be showing an ***n*** that is a multiple of 8. In this instance, there is a single largest rectangle, and that rectangle is a square connecting vertices ***k***/2 - 3***k***/2 - 5***k***/2 - 7***k***/2. Indeed, it is a ***k*** by ***k*** square, so the rectangles count from this square is k^2.

The *Largest Rectangles Excel* file. The table is structured in color-coded parts. The final three columns to the right in green provide information regarding the *slanted* ***n*** = 4***k*** images in rows 5-6 from the previous section that will be discussed at the end of this section. There are two main parts, each having three columns, ***j***, ***h***, and ***r***/2 = ***jh***, dealing with ***n*** = 4***k***+2, odd multiples of 2 images from rows 1-2, and ***n*** = 4***k***, even multiples of 2 images from rows 3-4. For each, as ***j*** increases by 1, ***h*** decreases by 2. These rows provide half ***r*** counts. The gold colored columns and cells provide full ***r*** counts. The table shows the results for ***k*** = 20, primarily because it provides some interesting patterns that become quite clear with larger ***k*** using first quadrant counts rather than half image counts for ***n*** = 4***k*** images. The first quadrant results show half ***h*** and quarter ***r*** counts with the first two for ***k*** = 20 the second two for ***k*** = 21 in four gray columns. Conclusions at the bottom detail how the largest ***r*** counts change for even and odd ***k*** for horizontal and slanted versions.

Excel aside. The entire file is controlled by ***k* in yellow in cell A4**. This cell has been renamed k since it makes writing equations very easy. (If you click on that cell, note that it initially says A4 but then says k in the Name Box in the upper left corner.) If you click on other cells, note that many are written as functions of k. Examples are provided in notes at bottom of the table for h, max r and w. (MOD is the remainder function and "" means show the cell as empty.)

For horizontal rectangles on even polygons images, the vertical diameter is a line of symmetry so the largest rectangle count is twice the right-side maximum, 0 < j ≤ k. Note the first quadrant values for n = 4k in gray are the diagonals of the times table.*										**n = 84, k = 21, 840 Rectangles Image, Static 1st Quadrant Values**		*Largest **n** = 4k Slanted Image Rectangles Counts for all Rectangles*		
Odd multiples of 2		**420** Max r		**Even multiples of 2**			**400** Max r							
n = 4k+2 =	82	210	Max r/2	n = 4k =	80		200	Max r/2	**760** Total in image					
20	k	10	Max j	20	k (=A4)		10	Max j				***399***	*Max r*	***n =***
h = 2k+1-2j > 0 else missing			**800**		h = 2k-2j > 0 else missing							*Total*	***761***	***80***
j, vertices 1-k (width)	h (height)	r/2 = jh (total/2)	In Image r	j (width)	h height	r/2 = jh	r	**1st Quadrant h/2**	**r/4**	**h/2**	**r/4**	*w = 2j-1*	*h*	*r = wh*
1	39	39	**78**	1	38	38	**76**	**19**	**19**	**20**	**20**	***1***	***39***	***39***
2	37	74	**148**	2	36	72	**144**	**18**	**36**	**19**	**38**	***3***	***37***	***111***
3	35	105	**210**	3	34	102	**204**	**17**	**51**	**18**	**54**	***5***	***35***	***175***
4	33	132	**264**	4	32	128	**256**	**16**	**64**	**17**	**68**	***7***	***33***	***231***
5	31	155	**310**	5	30	150	**300**	**15**	**75**	**16**	**80**	***9***	***31***	***279***
6	29	174	**348**	6	28	168	**336**	**14**	**84**	**15**	**90**	***11***	***29***	***319***
7	27	189	**378**	7	26	182	**364**	**13**	**91**	**14**	**98**	***13***	***27***	***351***
8	25	200	**400**	8	24	192	**384**	**12**	**96**	**13**	**104**	***15***	***25***	***375***
9	23	207	**414**	9	22	198	**396**	**11**	**99**	**12**	**108**	***17***	***23***	***391***
10	21	210	**420**	10	20	200	**400**	**10**	**100**	**11**	**110**	***19***	***21***	***399***
11	19	209	**418**	11	18	198	**396**	**9**	**99**	**10**	**110**	***21***	***19***	***399***
12	17	204	**408**	12	16	192	**384**	**8**	**96**	**9**	**108**	***23***	***17***	***391***
13	15	195	**390**	13	14	182	**364**	**7**	**91**	**8**	**104**	***25***	***15***	***375***
14	13	182	**364**	14	12	168	**336**	**6**	**84**	**7**	**98**	***27***	***13***	***351***
15	11	165	**330**	15	10	150	**300**	**5**	**75**	**6**	**90**	***29***	***11***	***319***
16	9	144	**288**	16	8	128	**256**	**4**	**64**	**5**	**80**	***31***	***9***	***279***
17	7	119	**238**	17	6	102	**204**	**3**	**51**	**4**	**68**	***33***	***7***	***231***
18	5	90	**180**	18	4	72	**144**	**2**	**36**	**3**	**54**	***35***	***5***	***175***
19	3	57	**114**	19	2	38	**76**	**1**	**19**	**2**	**38**	***37***	***3***	***111***
20	1	20	**40**	20						**1**	**20**	***39***	***1***	***39***
*Due to space considerations, variables are not ***bold italics***. Note: j+h/2 = k for n = 4k										*w=IF(E7<k+1,2*E7-1,"")*				
Row 7 h eqs.	h=IF(2*k+1-2*A7>0,2*k+1-2*A7,"")			h=IF(2*k-2*E7>0,2*k-2*E7,"")						*h=IF(2*k+1-2*E7>0,*				
When k is even, max is at j = k/2				When k is even, max is at j = k/2						*2*k+1-2*E7,"")*				
r/2 = k/2*(2k+1-2(k/2)) = k(k+1)/2				r/2 = k/2*(2k-2(k/2)) = k^2/2						*Even k, j=(k+1 ± 1)/2 are Max j*				
So Max r = jh = k(k+1) for even k.				So Max r = k^2 for even k.						*Odd k, j=(k+1)/2 is Max j*				
When k is odd, max is at j = (k+1)/2				When k is odd, two values j=(k ± 1)/2 are max.						*The Slanted n = 4k Max r*				
r/2=(k+1)/2*(2k+1-2((k+1)/2))=(k+1)k/2				r/2 = (k+1)/2*(2k-2((k+1)/2)) = (k+1)(k-1)/2						*Equation only differs from*				
So Max r = k(k+1) for odd k.				So Max r = (k+1)(k-1) for odd k.						*the Horizontal n = 4k Max r*				
Max r eqs.	Max r = k*(k+1) if n = 4k+2			Max r = IF(MOD(k,2)=**0**,k^2,(k+1)*(k-1)) if n = 4k						*Equation by MOD(k,2)=**1***				

Rows are populated with values if ***j*** is in the first quadrant, otherwise the row is empty. (In the ***k*** = 5 example, there are no values for ***j*** = 5 and with ***k*** = 20 there are no horizontal image values when ***j*** = 20 and more generally there are no ***j*** = ***k*** horizontal ***h***/2 counts if ***n*** is an even multiple of 2.) In both instances height is given by ***n***/2-2***j*** for 1 ≤ ***j*** ≤ ***k*** with the only difference being the fact that ***n***/2 = 2***k***+1 for odd multiples of 2 and 2***k*** for even multiples of 2. The pattern is clear. The size of the ***r***/2 rectangle rises then falls with largest rectangle near ***j*** = ***k***/2. Specifics are noted at the bottom of the table covering all possibilities.

Patterns of Decline from the Max *r* Value. When ***n*** is an odd multiple of 2 the decline from the Max ***r***/2 value is asymmetric but there is a notable pattern. On each side, the decline from the higher adjacent value is an odd number. Further if you bounce from side to side of Max ***j***, you get the first ***k*** odd numbers. When ***k*** = 5 we see a decline of 1 going from ***j*** = 3 to ***j*** = 2 but 3 going in the other direction. The ***r***/2 values from 1-5 were 9, 14, 15, 12, 5 so from the top the declines in bouncing back and forth were 1, 3, 5, 7, 9 by looking from ***j*** = 3 to 2 to 4 to 1 to 5 to 0. The same decline happens for other odd ***k*** starting from ***j*** = (***k***+1)/2 – the next largest rectangle is at (***k***-1)/2, the next smaller ***j***. Exactly the opposite occurs when ***k*** is even. The same decline pattern of increasing odd numbers occurs but the decline of 1 is now at the next larger ***j*** from the top, or ***j*** = ***k***/2+1 as we see with ***k*** = 20 where there is a decline of 1 = 210-209 in going from ***j*** = ***k***/2 = 10 to ***j*** = 11 and the decline of 3 = 210-207 in going to the next smaller ***j*** = ***k***/2-1 = 9.

By contrast, the decline on either side of the Max ***r*** value is symmetric when ***n*** is an even multiple of 2 but the pattern depends on whether ***k*** is even or odd. This pattern is further dissected by looking at half the ***r***/2 values.

A Pattern Straight out of a Multiplication Table. Since all heights are even for ***n*** = 4***k***, we can consider the first quadrant alone (then multiply by 4 for all quadrants). Notice what happens in this instance: As ***j*** increases by 1, ***h***/2 decreases by 1 and the sum remains ***j***+***h***/2 = ***k***. When ***k*** is even, the pattern is exactly the difference between squares pattern discussed here. This pattern covers half of all entries in a multiplication table. When ***k*** is odd, then the other half of the entries are covered as well. The only way to truly see this is to look at a multiplication table by diagonal (with slope of -1). The two diagonals shown in the table above are seen as the x + y = 20 and x + y = 21 diagonals in the multiplication table.

Slanted Largest Rectangles. The slanted ***n*** = 4***k*** images on shown in rows 5 and 6 of the previous section provide a nice counterpoint to the horizontal ***n*** = 4***k*** images in rows 4 and 5. We saw there that there is always one more rectangle in the slanted version: 761 versus 760 for ***k*** = 20 and ***n*** = 80 in the table. Beyond that things get more interesting.

We cannot slice the slanted image in half with a vertex to vertex diameter that is also a line of symmetry because the smallest rectangle span is 1 vertex in both directions, so we must resort to full rectangle counts rather than ***r***/2. The "tilted" height when ***j*** = 1 is from 1 to 2***k*** or 2***k***-1 rectangles with "tilted" width of 1 so for the ***k*** = 2 image in row 5 it is 3 rectangles high and for the ***k*** = 3 image it is 5 rectangles high. But what happens as we change ***j***? In this setting, we need to consider both how height and width change as ***j*** changes. Both change by 2 for each change in ***j***. As ***j*** increases, width increases by 2 starting at 1 and height decreases by 2 starting at 2***k***-1. Row 7 equations for ***w*** and ***h*** are provided in the green notes at the bottom of the table and you can see the resulting ***w*** and ***h*** values in the green columns to the right for ***k*** = 20. The product ***r*** = ***w·h*** is noted in the far right column in gold since it is full ***r*** rather than half.

Interestingly, the largest ***r*** equations for horizontal and slanted are incredibly close to one another. Both depend on whether ***k*** is even or odd. One is k^2 and the other is (***k***-1)(***k***+1) = k^2-1 due to the difference between squares formula. As we see from the table, the largest rectangle in the ***k*** = 20 horizontal image has 400 rectangles created by connecting vertices 10-30-50-70 of the 80-gon. There are two largest rectangles in the slanted 80-gon rectangles image, and both have 399 rectangles. One connects vertices 10-31-50-71 and the other connects vertices 11-30-51-70. The first is 19 wide by 21 high and the second is 21 wide by 19 high. (Both are rectangles because opposing vertices are 40 vertices, or ***n***/2 vertices from one another.)

Change to ***k*** = 19 and the reverse holds true. Now the largest slanted rectangle of the 76-gon is ***r*** = 361 = 19^2 connecting vertex ***j*** = (***k***+1)/2 = 10 to vertices 19 farther around the circle: 10-29-48-67. There are two largest horizontal rectangles having 360 rectangles: the 18 wide by 20 high rectangle 9-29-47-67; and the 20 wide by 18 high rectangle 10-28-48-66.

Therefore, as noted at the bottom of the table, the only difference between the two equations for Max ***r*** is the following:

For horizontal images, Max ***r*** = IF(MOD(***k***,2)=0,k^2,k^2-1);   For slanted images, Max ***r*** = IF(MOD(***k***,2)=1,k^2,k^2-1).

There is always a *k* by *k* square if *n* = 4*k*. If ***n*** = 4***k***, you can always find a ***k*** by ***k*** square sub-image based on a rectangles image of that ***n***-gon. If ***k*** is even, that image is horizontal; if odd, it is slanted. Thus, for ***k*** = 4, the largest square connects vertices 2-6-10-14 in row 3; for ***k*** = 5, the largest square connects vertices 3-8-13-18 in row 5 of the previous section.