Some Geometric Observations about 0-5-7 Images

The counting exercise shown in the last section was simply suggestive of other exercises you could undertake, or suggest that students undertake as they work collectively to answer to the question … how many triangles are there in an image?

It is worth drilling down a bit on the 0-5-7 example to see the kinds of patterns to look for when you change values, because, of course, the patterns will change as well. The point to consider is how they change and why.

Vertex Patterns. There are several patterns that emerge by staring at the images in the previous section.

1. Since the 3rd line connects 5-7, there is no line in that direction using vertex 6. Thus, 6 is ALWAYS a 2 line vertex.
2. When ***n*** is even, there is always a second 2 line vertex at 6+***n***/2.
3. More generally, when ***n*** is even, there is 180° rotational symmetry in the image.
4. When ***n*** is odd, there is no rotational symmetry (but ***n*** = 9 has a line of symmetry on the vertex 7 diameter).
5. There is always a parallelogram with one corner at 0 and the other at 6.
    a. Two of the corners of this parallelogram are vertices 0 and 6 and the other two are interior for ***n*** > 8.
    b. The angles at these two vertex corners span 2 vertices. One is angle 5-0-7, the other is 1-6-(***n***-1).
    c. The other two corners are on the diameter from vertex 3-(3+***n***/2) regardless of whether ***n*** is even or odd.
        i. If ***n*** is even, the other side (3+***n***/2) is also a vertex of the ***n***-gon but it is between vertices if ***n*** is odd.
6. Vertex 3 is always a zero-count vertex. This is the only zero-count vertex on the right side of the image.
    a. From vertex 3 in both directions (towards 2, 1, 0 and 4, 5, 6), apex counts increase from **1** to **3** to **5**.
    b. Vertex 0 and vertex 6 are the first vertices with maximal apex counts of **5** looking counterclockwise and clockwise from vertex 3.
7. When ***n*** is even, the other zero-count vertex is vertex 3+***n***/2 at the other end of the diameter starting at 3.
8. When ***n*** is odd, there are two zero-count vertices opposite 3.
    a. They are at vertex 3+(***n***-1)/2 and at vertex 3+(***n***+1)/2.
    b. They are also the other two vertices with only two lines (since ***n*** odd means there is one odd-vertex-out in each direction).
    c. These vertices create a bowtie between vertices with apex triangles count of **2** at the knot of the bowtie. This is the last interior apex triangles count looking from right to left.
9. There is only one set of interior apex intersections because the smallest angle is 2.
    a. This is easiest to see for ***n*** > 9.
        i. For ***n*** = 8, the 0-5-7 angle spans a single vertex and is therefore sharper than 2.
        ii. For ***n*** = 9 it spans 2 vertices so that second-sharpest base angle isosceles triangles result. Use the obtuse apex angle of the isosceles triangles for counting rather than the base angle like we did here.
    b. More generally, if the smallest angle spans ***b*** vertices, then there will be ***b***-1 sets of interior intersections.
        i. As we will see, these intersections are on interior arcs rather than on lines.
10. Finally, it is worth noting that the ***n*** = 12 image is an example of sharpest isosceles triangles on even ***n***.

Why don't the apex edge counts increase beyond 5? Consider what happens in an image with ***n*** ≥ 14 as we move from vertex 0 to vertex ***n***-1. The apex at 0 has bases that are parallel to 5-7 and there are 5 such bases starting at vertices 1 to 5. But move to vertex ***n***-1 and the line 0-12 becomes the base of a new smallest triangle. At the same time, the base from 5-7 is lost since the largest triangle with apex at vertex ***n***-1 has the 5 vertex span corner at vertex 8 with base 8-4. Note that at ***n*** = 16, when we move from vertex 15 to 14 the same thing happens again. Now the new smallest triangle with apex at 14 has base 15-13 but 8-4 is lost and the largest 5 vertex span corner is at vertex 9 with base 9-3. For larger and larger ***n***, this process continues. Every time a new base is gained at the top, an old one is lost at the bottom because the angle at that vertex spans 5 vertices, so the sum remains **5**. This continues until vertex 6+(***n***+1)/2 for odd ***n*** beyond which there is a decline from **4** to **2** to **0**, or vertex 6+***n***/2 for even ***n*** after which the apex count declines from **3** to **1** to **0**.

The interior apex counts never gets larger than **6** for the same reason. And the reason for **6** rather than **5** is that if you look at the size of the triangles at the base, they need only span a single vertex on either side of the apex when counting from the interior, but they must span two vertices when counting from the perimeter of the ***n***-gon.