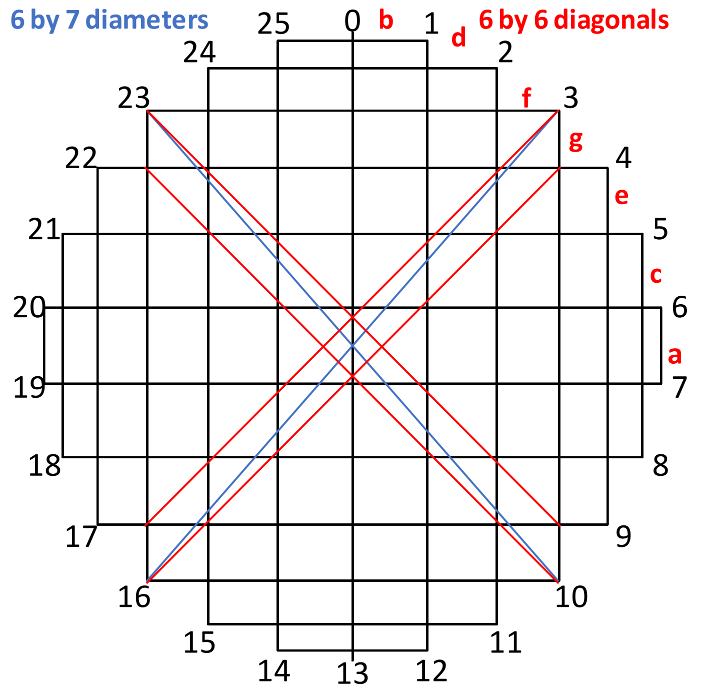
Squares in Rectangles Images

When rectangles have equal length sides, the result is a square. As noted in the discussion of rectangles in a clockface, one way to distinguish squares is that at least one of the two opposing sets of corners of the rectangle lie on a diameter of the circle. If both sets of opposing corners lie on diameters of the circle the square is centrally located.

None of the rectangles are squares when *n* = 4*k*+2. The image at right is the ***k*** = 6, ***n*** = 26 odd multiple of 2 rectangles image. **Blue diameters** connect opposing vertices of the largest rectangle 3-10-16-23 which is 6 vertices wide by 7 high. These diameters intersect at the center of the circle. There are two sets of **red diagonals** connecting opposing vertices of the two possible 6 by 6 rectangles.



About lengths of sides. The letters **a** – **g** represent the lengths of horizontal and vertical one rectangle wide widths for **b**, **d**, and **f**, and heights for **a**, **c**, **e**, and **g** based on vertices 0-7 of the 26-gon. [**MA.** Actual locations of those points are easily obtained using sine and cosine functions. From there, it is just a matter of addition and subtraction to find side lengths.] The table beneath summarizes this information.

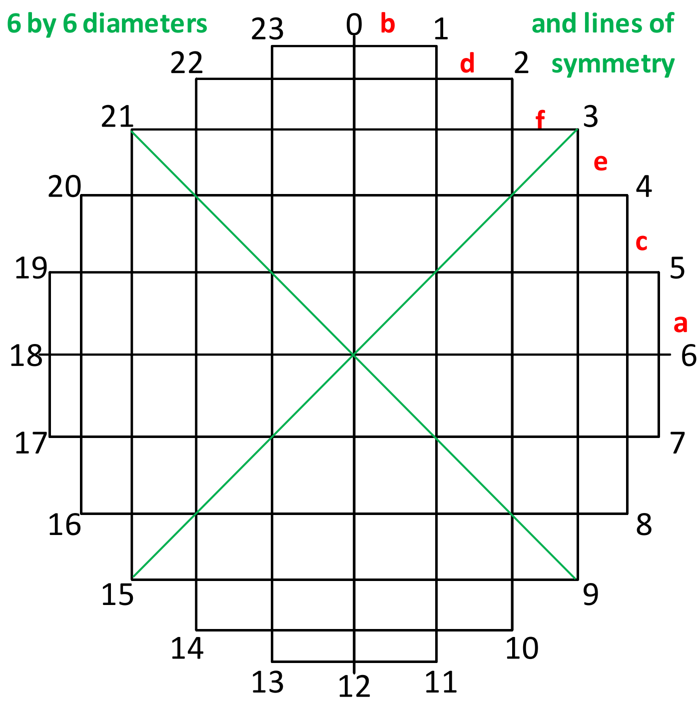
The distance between adjacent vertices of a regular polygon is the same for a given ***n***, that is what makes it regular. For an ***n*** = 4*k*+2 polygon, that distance is the same as twice the y coordinate of vertex ***k*** since vertex ***k*** and ***k***+1 have the same x coordinate and the y coordinate of ***k***+1 is simply the opposite of the y coordinate of ***k*** (as we see for vertices 6 and 7 in the table for ***k*** = 6 so ***n*** = 26). Every other horizontal or vertical distance can be considered as a leg or side of a right triangle whose hypotenuse is this distance, noted as **a** in the figure and table. Each leg is smaller than the hypotenuse.

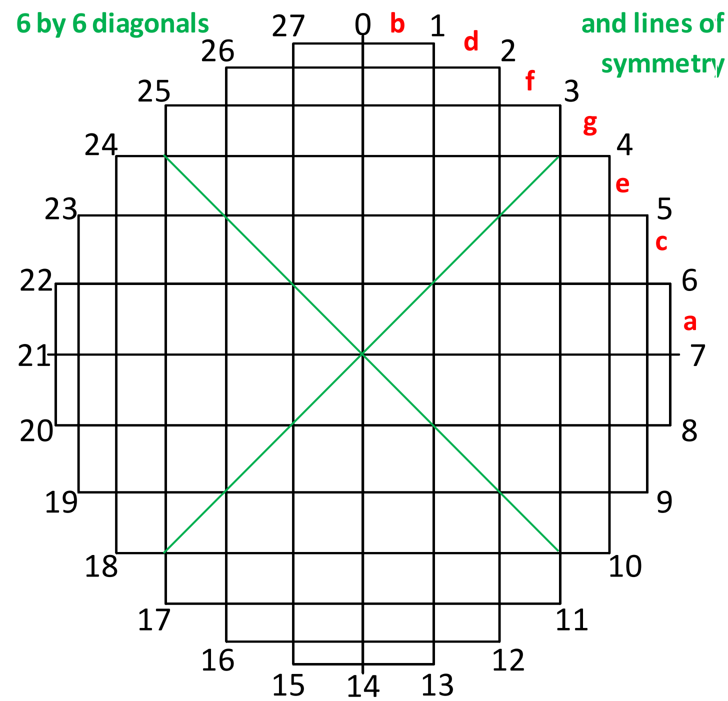
26-gon vertices in 1st quadrant*			Change from adjacent (**letters**)			
j	x	y	x (***j***-(***j***-1))		y ((***j***- 1)-***j***)	
0	6.126E-17	1				
1	0.2393157	0.9709418	**0.2393157**	**b**	0.0290582	
2	0.4647232	0.885456	**0.2254075**	**d**	0.0854858	
3	0.6631227	0.7485107	**0.1983995**	**f**	0.1369453	
4	0.8229839	0.5680647	0.1598612		**0.180446**	**g**
5	0.9350162	0.3546049	0.1120324		**0.2134599**	**e**
6	0.9927089	0.1205367	0.0576926		**0.2340682**	**c**
7*	0.9927089	-0.120537	0		**0.2410734**	**a**
*7th point is in 4th quadrant			Horizontal		Vertical	
			Distance between Lines			
The x and y data is drawn from the Polygon Vertices sheet of *General Triangles* file setting A1 = 26.						
Dimensions of the 6 by 6 rectangle whose upper right corner is ***j*** = 3.						
upper	width =	1.3262453	= 2*x for ***j*** = 3.			
red cross	height =	1.3165755	= y for ***j*** = 3 plus y for ***j*** = 4.			
The lower 6 by 6 rectangle (lower right at ***j*** = 10) has the same dimensions. This is the rectangle with the **lower red cross**.						

We know that as the angle increases, the size of the leg opposing that angle increases and the size of adjacent side decreases. This is why we see the progression of lengths in the table from largest to smallest of **a** > **b** > **c** > **d** > **e** > **f** > **g**. Since they are never equal, none of the smallest rectangles are squares. The bottom of the table shows the calculation of length of the 6 by 6 rectangles with **red cross diagonals**. Both are a bit wider than they are high.

Note in the table that x is smaller than y for ***j*** = 0 to 3 but larger for ***j*** = 4 to 6. This is true more generally until the angle is 45°, at which point the two lengths are the same.

When ***n*** = 4***k***+2 there is no vertex at ***n***/4 so the deviations from horizontal and vertical are not symmetric. But if ***n*** = 4***k***, the deviations are symmetric meaning that a horizontal distance in going from 0 to 1 is the same as the vertical distance in going from ***k*** to ***k***-1, and so on. This follows from the fact that ***n*** = 4***k*** rectangles images have 90° rotational symmetry but ***n*** = 4***k***+2 rectangles images do not have 90° rotational symmetry. The next two figures examine this symmetry for an even and an odd value of ***k*** as the images alternate between having the square on the ***n*** = 4***k*** frame or interior to it.





24-gon vertices in 1st quadrant*			Change from adjacent (**letters**)				28-gon vertices in 1st quadrant*			Change from adjacent (**letters**)			
j	x	y	x (***j*** -(***j*** -1))		y ((***j***- 1)-***j***)		***j***	x	y	x (***j*** -(***j*** -1))		y ((***j***- 1)-***j***)	
0	6.126E-17	1					0	6.126E-17	1				
1	0.258819	0.9659258	**0.258819**	**b**	0.0340742		1	0.2225209	0.9749279	**0.2225209**	**b**	0.0250721	
2	0.5	0.8660254	**0.241181**	**d**	0.0999004		2	0.4338837	0.9009689	**0.2113628**	**d**	0.073959	
3	0.7071068	0.7071068	**0.2071068**	**f**	0.1589186		3	0.6234898	0.7818315	**0.1896061**	**f**	0.1191374	
4	0.8660254	0.5	0.1589186		**0.2071068**	**e**	4	0.7818315	0.6234898	**0.1583417**	**g**	**0.1583417**	**g**
5	0.9659258	0.258819	0.0999004		**0.241181**	**c**	5	0.9009689	0.4338837	0.1191374		**0.1896061**	**e**
6	1	0	0.0340742		**0.258819**	**a**	6	0.9749279	0.2225209	0.073959		**0.2113628**	**c**
			Horizontal		Vertical		7	1	0	0.0250721		**0.2225209**	**a**
			Distance between Lines							Horizontal		Vertical	
Note the symmetry between x and y about ***k*** meaning					$x_j = y_{k-j}$ and $y_j = x_{k-j}$ for $j \le k$.					Distance between Lines			
In each ***n*** = 4***k*** image, this symmetry leads to **a** = **b**, **c** = **d**, **e** = **f**, and so on. When ***k*** is even the largest square connects vertices ***k***/2-3***k***/2-5***k***/2-7***k***/2.													

As noted at the bottom of the table, the vertical and horizonal distances between adjacent vertices leads to pairwise equalities and hence internal squares in both images. None of the smallest internal squares are central squares meaning that both opposing corners are on the **two green diagonals** (when ***k*** is even, these diagonals are also diameters, but when ***k*** is odd, these diagonals can be extended to diameters whose endpoints are midway between vertices). In both images, there are 2×2, 4×4 and 6×6 central squares. All other squares have one, but not both, sets of opposing corners on one of the two **green diagonals**.

When ***k*** is even, such as ***k*** = 6 to the left, ***n***/8 is a vertex of the ***n***-gon, and hence it provides the upper right corner of the largest rectangle as discussed in the last section. In this instance, it is also a ***k*** by ***k*** square. However, when ***k*** is odd, one can write ***k*** as ***k*** = 2***m***+1 and we saw that there were two largest rectangles in the image. One has upper right vertex ***m*** the other has upper right vertex ***m***+1. Both are 2***m*** by 2***m***+2 sized rectangles. The intersection of the two rectangles is 2***m*** by 2***m*** square such as **the 6 by 6 square whose diagonals are shown in green** in the right image above, given ***m*** = 3 or ***k*** = 2***m***+1 = 7 so ***n*** = 4***k*** = 28.

We see that ***n*** = 4***k*** horizontal rectangles images have a largest square that is always even and increases in size on even values of ***k***. If you analyze the slanted ***n*** = 4***k*** rectangles images, you will find that the largest square is always odd, but it increases on odd values of ***k***. This is easy to see by looking at ***k*** = 2, 3, 4 in row 5 of An Introduction to Rectangles.