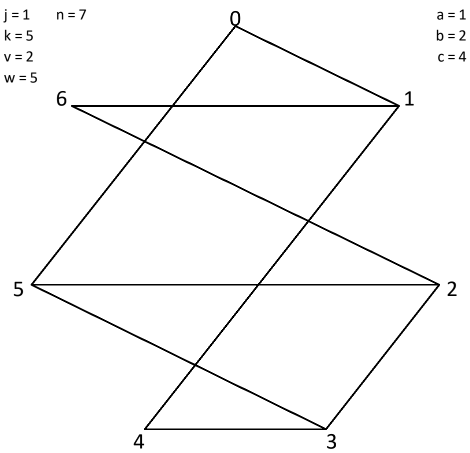
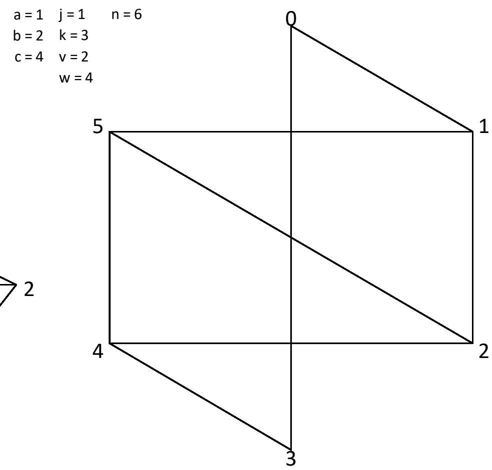
Conditions under which there is a Vertex Triangle in a Triangles Image

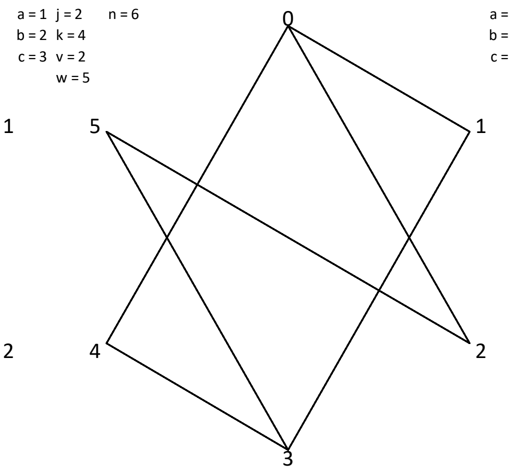
Triangles have 3 corners, but triangles in a triangles model rarely have all three corners as vertices of the ***n***-gon. We call such a triangle a *vertex triangle*.

Definition. A *vertex triangle* is a triangle whose corners are vertices of the underlying ***n***-gon on which the triangles image was created.

Some Examples. The first image has one vertex triangle, 2-3-5; the second has two, 1-2-5 and 2-4-5; and the third has none, despite having the same ***a***, ***b***, ***c*** of 1, 2, 3 or 30-60-90° as the second since ***n*** = 6 for both images.



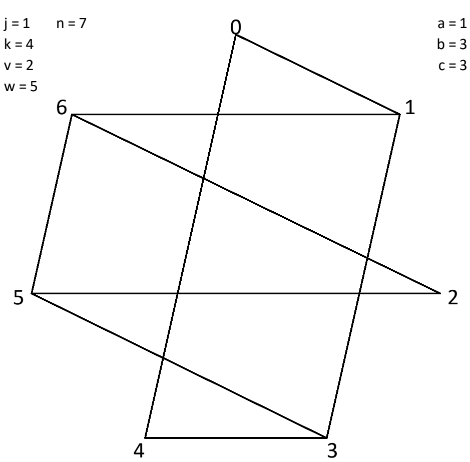




A Special Case. We can guarantee that a triangles image has a vertex triangle by setting ***v*** = ***j*** and ***w*** = ***k*** (or *vice versa*) since the resulting image will include triangle 0-***j***-***k***. But if we do not include that restriction on ***v*** and ***w*** (like we did with the images above), then when will the image include a vertex triangle? Additionally, if there is one, when is there more than one (like we saw with the second image above)?

Think Angles but Focus on Sides. The above images were chosen because there is a small, medium and large angle which translates into a small, medium and large side opposite those angles. The sum of angles ***a***+***b***+***c*** = ***n*** and if the triangle in question is a vertex triangle, then the side-lengths span ***a***, ***b***, and ***c*** vertices as well. The ***a*** = 1 span side is 2-3 in the left image and 1-2 and 4-5 in the right image while the ***b*** = 2 span is 3-5 in the left and 2-4 and 5-1 in the right. In the middle image, the ***c*** = 3 span, the hypotenuse of both right triangles is the diameter 2-5. The span 5-2 in the left image spans 4 vertices (5-6-0-1-2) that are opposite angle 2-3-5 or ***c*** = 4.

It is worth noting that the sides are related to the angles opposite the side so that the same length is associated with two different angles. The left image below also based on ***n*** = 7, has a vertex triangle at 2-5-6 and the span 5-2 spans 3 vertices (2-3-4-5) that is opposite angle 2-6-5 or ***b*** or ***c*** = 3 since the triangles in this image are isosceles. This same point could be made for any triangles image. Any side of length 1 < ***d*** ≤ ***n***/2 can be associated with two angles, ***d*** and ***n***-***d***.



The angles are supplementary angles (summing to ***n*** or 180°). Unless both are right angles, one is acute, and the other is obtuse. If the angle ***n***-***d*** is obtuse (like ***c*** = 4 to the left above) the actual side length is the same as its acute ***n***-***c*** counterpart (like 7-4 = ***b*** = ***c*** = 3 at the lower left with lengths 2-5 and 2-6).

Even and Odd Side Lengths. We saw elsewhere that the number of lines in a triangles image depends on whether ***n*** is even or odd. When ***n*** is odd, there are (***n***-1)/2 lines in each direction and each is a different length, spanning from 1 to (***n***-1)/2 vertices. These lengths are even numbers on one side, and odd numbers on the other side (to take one direction, there are odd spans of 1 and 3 from lines 0-1 and 6-2 starting from 0.5 in the lower left image and even span of 2 from 3-5 starting from vertex 4 on the other side).

When n is even, the situation is more complex. There are not always the same number of lines per direction. Consider the two n = 6 images. the middle image has $n/2-1 = 2$ horizontal, $n/2 = 3$ vertical and 3 slanted in the 0-1 direction, but the right has 2 lines in two directions (0-2 and 0-4) and 3 lines in one direction (0-1). When there are $n/2-1$ lines in a direction, each line spans an even number of vertices, but when there are $n/2$ lines, each line spans an odd number of vertices.

The Reason the Right Image has no Vertex Triangle. To have a vertex triangle, you require spans of a, b, and c between vertices. The right image angles are a = 1, b = 2 and c = 3, or two odd and one even number. We just saw that there were two directions where vertex spans are even and one where vertex spans are odd. This means that it is impossible to use the lines of the existing image and find three vertices spanning a, b, and c that are also lines of the image.

The Reason the Middle Image has Two Vertex Triangles. The middle image has two vertex span directions that are odd and one that is even and hence a vertex triangle can be found in this situation. Indeed, there are two. The second triangle is the same size as the first, the only difference is that this one is upside down. More specifically, if the vertex triangle vertices are v_1-v_2-v_3 then MOD($v_1+n/2,n$)-MOD($v_2+n/2,n$)-MOD($v_3+n/2,n$) will also be a vertex triangle in this image. This follows from the 180° rotational symmetry of all even triangles images.

The Reason the Left Images have One Vertex Triangle. With odd n images, each direction has both even and odd vertex spans. As a result, there will be a vertex triple that is a vertex triangle given odd n. The easiest way to find that triangle is to look at the apex angle, a. The apex angle is typically easier to see than the other angles. Next focus attention on the direction that forms the base for this angle. For both left images, a = 1 so look for a base that spans a single vertex opposite a. In the upper image, that span is 2-3, in the lower it is 5-6.

Challenge Questions. The images below share common a and b by row. Use the strategies described above to find the vertex triangles (if they exist) for each image.