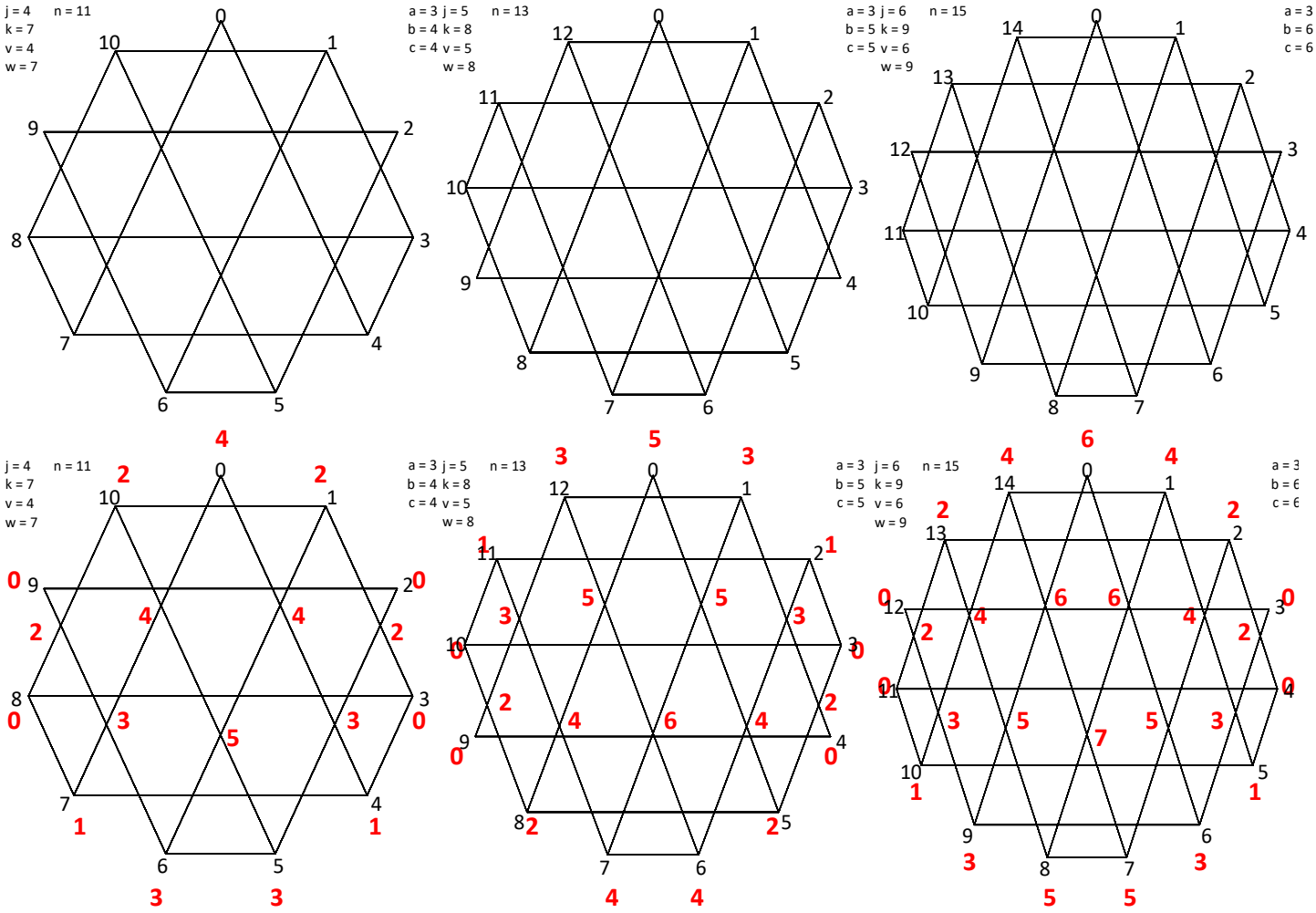
Odd Isosceles Triangles

The simplest counting exercise was sharpest isosceles triangles on odd polygons. As we argue in the first half of P3, the image based on $n = 2k+1$ using parallel lines connecting vertices 0 to k to $k+1$ has k^2 triangles according to the Hill Formula. This is seen by zig-zag summing alternating top and bottom halves. In the process, apex counts vary from 1+2+3+…+k-1+k+k-1+…+2+1. If we relax $a = 1$ but maintain odd isosceles images, a must be odd. We start with $a = 3$.

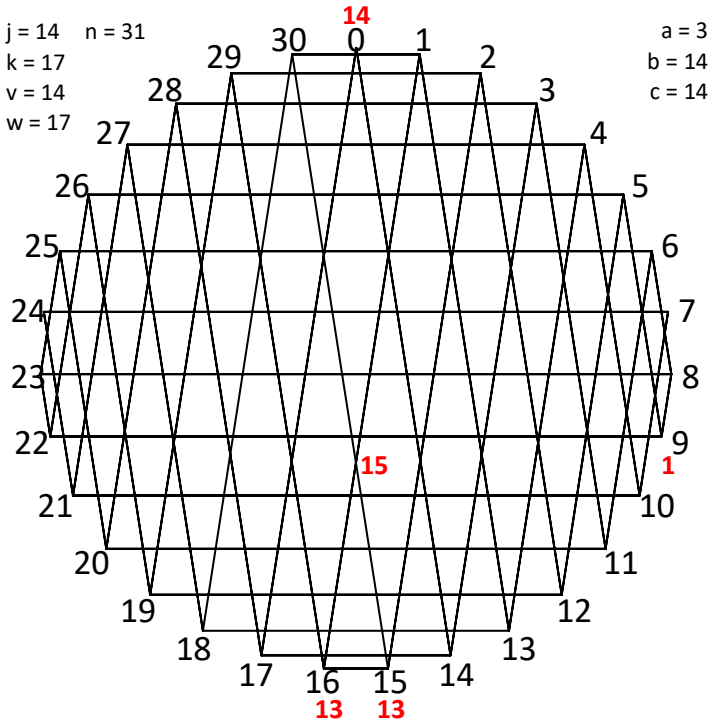
$a = 3$. When the smallest angle is larger than one then there are interior apexes on a-1 arcs (or 2 given $a = 3$) in addition to those on the perimeter of the image. Rather than focusing on these arcs individually, consider them as a pair.

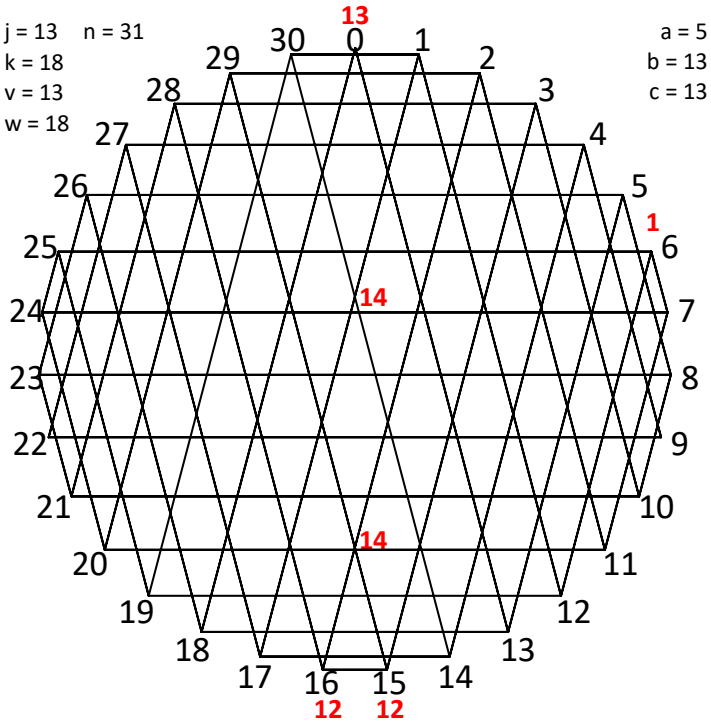


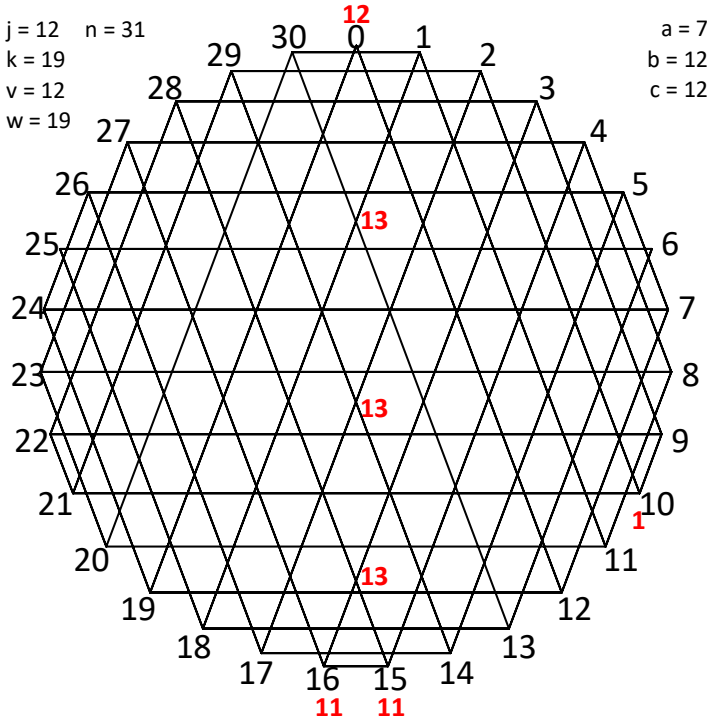
Odd Isosceles Triangles Image connecting vertices 0 - b - $a+b$ with angles a and b			Total Triangles Counts based on n and a with $b = (n-a)/2$.													
9	= a	angle at 0	a\n	5	7	9	11	13	15	17	19	21	23	25	27	29	31
11	= b	base angle	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225
			3	3	11	23	39	59	83	111	143	179	219	263	311	363	419
			5		5	18	37	62	93	130	173	222	277	338	405	478	557
	31	= $n = a+2b$ given isosceles Δ.	7			7	25	51	85	127	177	235	301	375	457	547	645
Interior Counts			9				9	32	65	108	161	224	297	380	473	576	689
a -1	8	Interior arcs (even, given odd a)	11					11	39	79	131	195	271	359	459	571	695
(a -1)/2	4	= d = Pairs of arcs	13						13	46	93	154	229	318	421	538	669
(b +1)^2-2	142	= f = Sum per pair of arcs	15		Black: $a < b$.					15	53	107	177	263	365	483	617
			17		Green: $a > b$						17	60	121	200	297	412	545
121	= b ^2	Exterior (n -gon apex counts)	19		Red: $a = b$, equilateral Δ.							19	67	135	223	331	459
568	= d * f	Interior apex counts	21		First 3 images.								21	74	149	246	365
689	**Total Triangles in Image**		23		Last 4 images, a =9, b = 11 at									23	81	163	269
			25		left is bottom right image.										25	88	177
To see an image in the *General Triangles Model*,			27		These are largest triangles count given n.											27	95
set $j = v = b$; $k = w = a+b$; and $n = a+2*b$.			29		1st row is *Sharpest Odd Isosceles Triangles.*												29

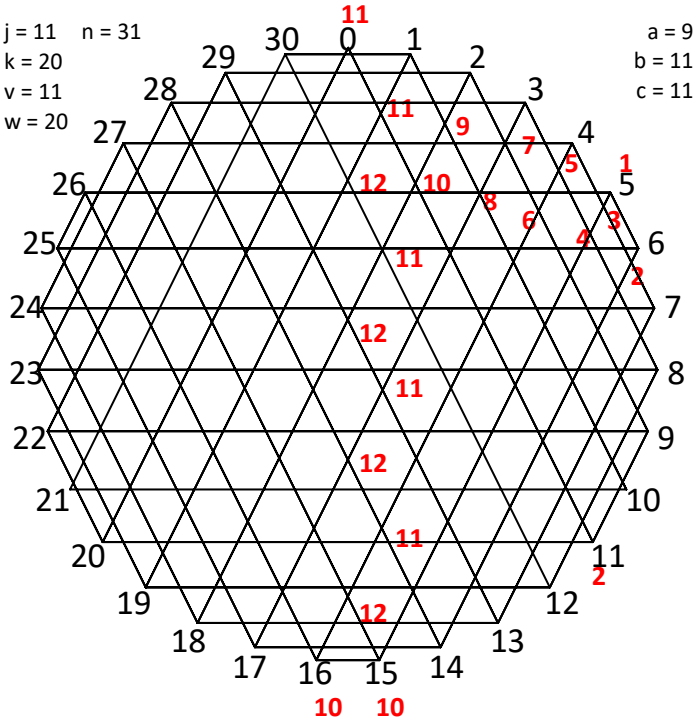
Consider first the exterior apex counts. The pattern from when ***a*** = 1 is seen in modified form here. The *Hill Formula* still applies with the top of the hill at vertex 0 having value (***n***-3)/2 = ***b***. As before the counts move from side to side but no longer follow the image's zig-zag lines. As noted elsewhere, interior apexes have no values lower than 2. But, by moving back and forth across both arcs, all values from 2 to ***b***+1 and back to 2 are included; the interior sum is (***b***+1)^2-2 and the total number of triangles given second sharpest odd isosceles triangles is T(***n***) = ((***n***-3)/2)^2+((***n***-1)/2)^2-2. *Notice that the largest internal and external apex counts are both on the vertical diameter of each image.*

General odd isosceles triangles. When ***a*** > 1 and odd there are (***a***-1)/2 internal apexes on the vertical diameter. The figures below show ***a*** = 3, 5, 7, and 9 for ***n*** = 31. Rather than provide all apex counts, only those on the vertical diameter from 0-15.5 and at **1** on the right hand side are shown, except the ***a*** = 9 version which shows the right half of apex counts for the top two arcs. Notice that the top arc has odd values from **3**-**11** and the second has even values from **2**-**12** = ***b***+**1**.







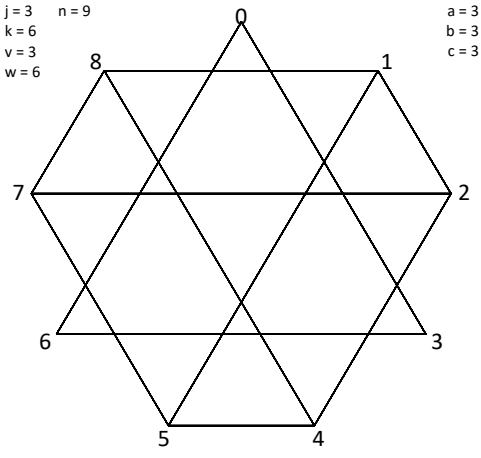


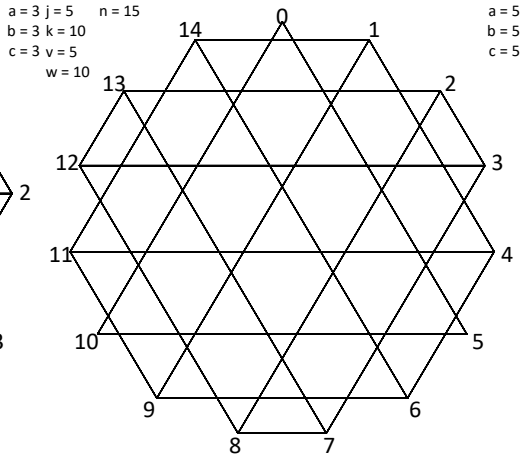
When viewed in pairs, the internal apex arcs range from **2** to ***b*+1** and back to **2**, regardless of how many pairs are shown.

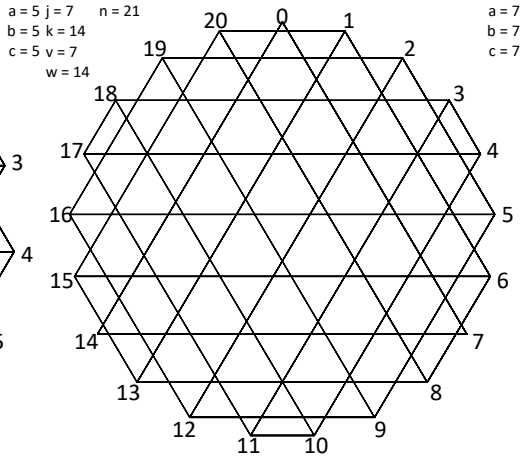
The left part of the table shows how the total triangles for $a = 9$, $b = 11$ so $n = 31$ is calculated. The total apex counts from the 31-gon vertices is $b^2 = 11^2 = 121$ and there are $(a-1)/2 = 4$ pairs of internal apex arcs, each with $(b+1)^2-2 = 12^2-2 = 142$ triangles so the total triangles in the image is $689 = 121+4\cdot142$.

We have followed the general strategy of counting using the sharpest angle as the distinguished point of triangular images. This need not be the case; it was simply used because it is often the easiest part of a triangle to see.

The same counting strategy occurs when the base angle of the isosceles triangle is equal to or smaller than the apex angle. Four cases when $a = b$ are shown in the table in **red**. These result in the equilateral triangles images shown below. The third image, $n = 21$, has 3 apparent concurrence points. A quick check shows that none are points of concurrence.



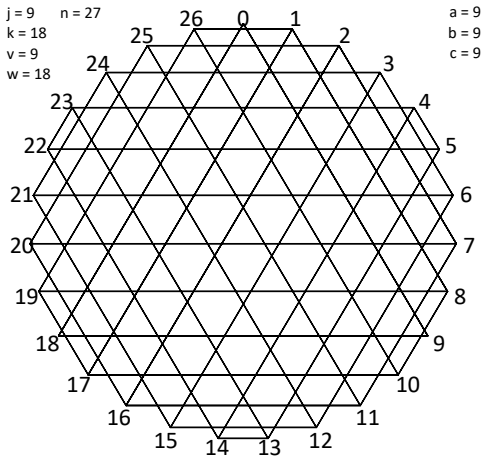




The $a > b$ count values are shown in **green**. *The General Triangles Model* will no longer call this angle a (because that model orders angles from smallest to largest). Nonetheless, the strategy set forth at the bottom left of the table will produce an angle of a at 0 (from vertices b-0-a+b) and b at vertices b and a+b.

The values highlighted in yellow show the odd angle a which produces the largest number of triangles given odd $n \geq a-2$. Note that in each instance, these values are equilateral or close to equilateral. As shown below, we can prove that this must be the case but to do so we need to use a bit of calculus.

The last entry in each column of the table has $a = n-2$ and $b = 1$. This is the most obtuse isosceles triangle. It ends up with the same result described earlier, albeit using a different approach.



Total Triangles as a Single Equation. The left part of the table provides the most systematic way to view total isosceles triangles on odd n as a function of odd apex angle $a \leq n-2$ but we can write this as a single equation given $b = (n-a)/2$:

$$T(a; n) = ((n-a)/2)^2 + ((a-1)/2)\cdot(((n-a)/2+1)^2-2).$$

MA. Obtaining the maximum a as a function of n. If T is maximized, then so is 8T (multiplying by 8 removes fractions). Upon simplification we have:

$$8T(a; n) = 2(n-a)^2 + (a-1)\cdot(n-a+2)^2 - 8a + 8.$$

Set the derivative equal to zero and regroup to obtain a quadratic in a:

$$d8T/da = 3a^2 - (6+4n)a + n^2 + 2n = 0.$$

	27	n	Maximum a	
A	$B = -(6+4n)$	$C = n^2+2n$	a =	9
3	-114	783		
Where a = QF = $(-B-(B^2-4AC)^{0.5})/(2A)$				

The *Excel* scrap at right shows that the solution for $n = 27$ is indeed $a = 9$ by applying the quadratic formula. When $n = 29$, the QF is 9.67 which is closest to odd $a = 9$ and when $n = 31$, the QF is 10.3 which is closest to odd $a = 11$.