On Maximum Apex Counts

It is worth noting that, as we do often when discussing triangles images, it is convenient to conceptualize angles in terms of the number of vertices spanned, rather than number of degrees since all angles created by these sets of parallel lines are multiples of 180/***n***° by the inscribed angle theorem. The vertices spanned in the *General Triangles* model are noted as ***a***, ***b***, and ***c*** where ***a***+***b***+***c*** = ***n***.

It is easy to see that the maximum apex count on the perimeter of an ***a*** ≤ ***b*** ≤ ***c*** triangles image is ***b*** by noting that one way we create an ***a, b, c*** image in the *General Triangles* model is to set ***J*** = ***V*** = ***b***, ***K*** = ***W*** = ***a***+***b***. This produces an image with a vertex triangle and works for any odd ***n*** = ***a***+***b***+***c*** and half of even ***n*** images. For even ***n*** images without a vertex triangle, set ***J*** = ***b***+*1*, ***V*** = ***b*** and ***K*** = ***W*** = ***a***+***b***+*1*. The top row on the next page shows the first version, and the bottom row shows the second. Any triangles image with angles ***a***, ***b***, and ***c***, is one of these two images up to a rotation or reflection.

The images on the next page are scalene triangles with ***a*** < ***b*** < ***c***. It is convenient to consider the directions as colors. **Red** is the direction opposite apex angle ***a***, **blue** opposite the middle angle ***b***, and **green** is opposite the large angle ***c***. Note that the apex angle is created by the **blue**/**green** intersection.

Perimeter Apex Counts. Both versions have apex count of ***b*** at vertex 0 because there are **red** bases associated with each vertex from 1 to ***b***. Apex counts decline as you move clockwise from 0. If you examine the top row on the next page, you will see that the apex count for vertex 1 (from 1 to ***b***-1) declines by 2 from ***b*** because a base is lost at the top and at the bottom. However, the initial decline in apex count is 1 at vertex 1 (from 1 to b) because a base is lost at the top but not the bottom. Subsequent moves (to 2, …) decline by 2 each time until either 1 or 2 remain at the right. After an apex count of 1 or 2 is achieved, there are between 1 and 3 apex counts of 0 before counts (of inverted triangles) increase in the bottom half of image.

If you move counterclockwise (to ***n***-1, …) you gain a base at the top but lose one at the bottom (unless ***b*** = ***c***) so that the apex count remains at ***b*** before it starts to decline at the left side. There need not be symmetry from side to side.

Interior Apex Counts. The second column shows an arbitrary interior ***a***, ***b***, ***c*** angled triangle which has an apex count of ***b***+1. The rationale behind this assertion is presented in the second and third columns. All intersections with the circle containing vertices of the regular ***n***-gon are vertices of the ***n***-gon. Put another way, all parameters ***e***, ***f***, ***g***, ***h***, ***p***, and ***q*** are whole numbers representing number of vertices contained in that arc. These adjacent arcs sum to ***n*** and each end between adjacent arcs represent vertices of the ***n***-gon. Because of the interior angle theorem, we know the size of the sum of opposing arcs: ***e***+***h*** = ***a***, ***f***+***p*** = ***b*** and ***g***+***q*** = ***c***.

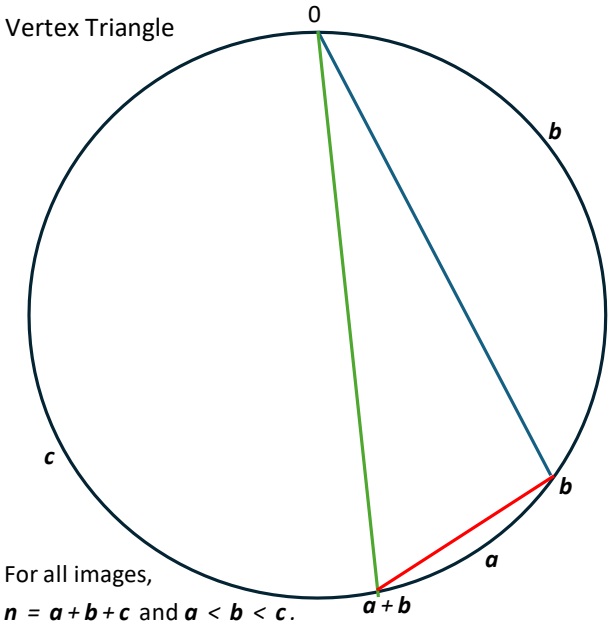
As drawn, ***g*** > ***p***, so the **red** bottom base separates ***g*** into two pieces, ***p*** and ***g***-***p*** and the top **red** base at the ***e*** / ***f*** border. This means there are **red** bases for intersecting the **green** and **blue** legs at ***f***+***p*** vertices and at the top vertex but note that ***f***+***p*** = ***b*** so there are ***b***+1 bases using the interior intersection point as apex.

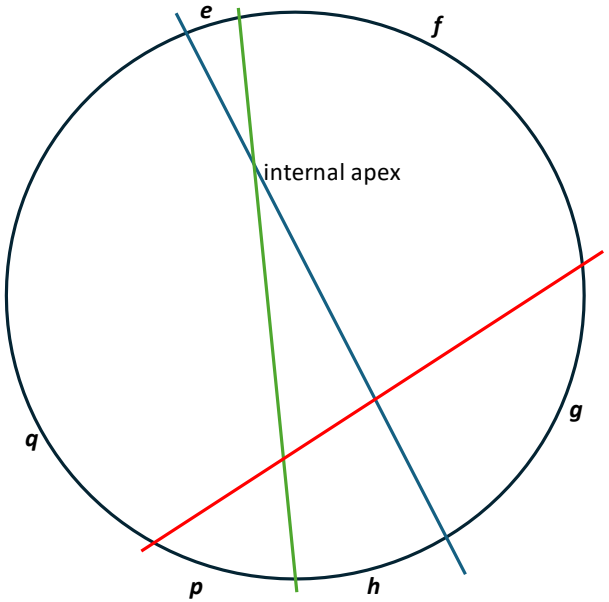
If ***g*** = ***p***, the bottom base would intersect at the ***h*** / ***g*** border there are still ***b***+1 bases at the interior apex. Finally, if ***g*** < ***p***, then the bottom **red** base would be at the ***h*** / ***g*** border and there are ***f***+***g***+*1* < ***f***+***p***+*1* = ***b***+1 bases at the interior apex.

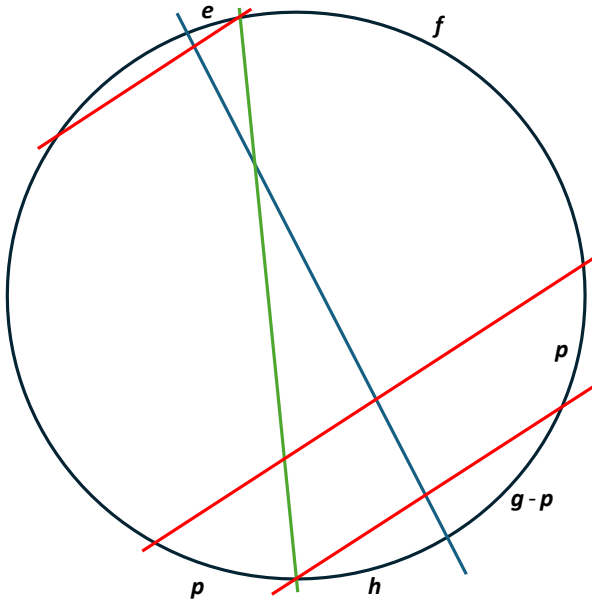
If the interior apex is in the left part of the interior, then we could create images where the top red base uses the vertex at the ***e*** / ***q*** rather than the ***e*** / ***f*** border. This produces an image count smaller than ***b***+1 just like the ***g*** < ***p*** case just discussed.

As discussed elsewhere, if the interior apex is also a point of concurrence, the apex count is one smaller than the above analysis would suggest.

Vertex Triangle



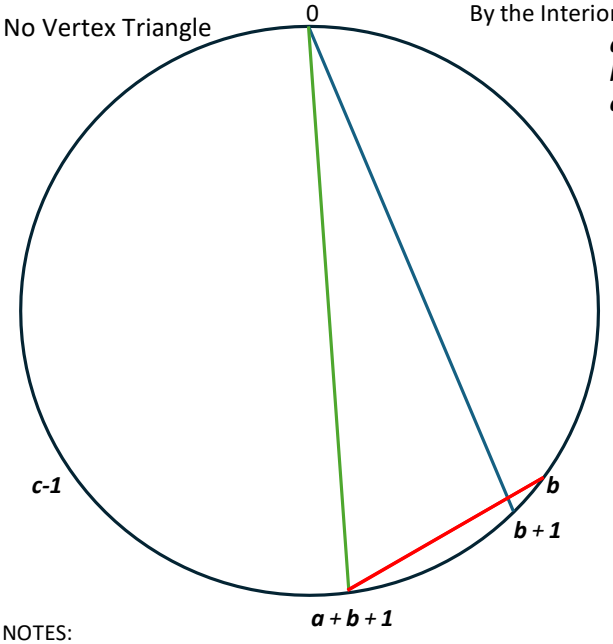




For all images,

$n = a + b + c$ and $a < b < c$.

No Vertex Triangle

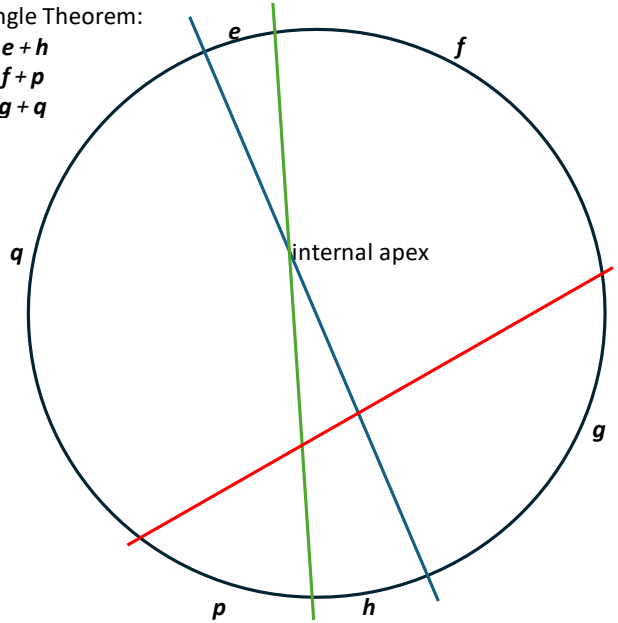


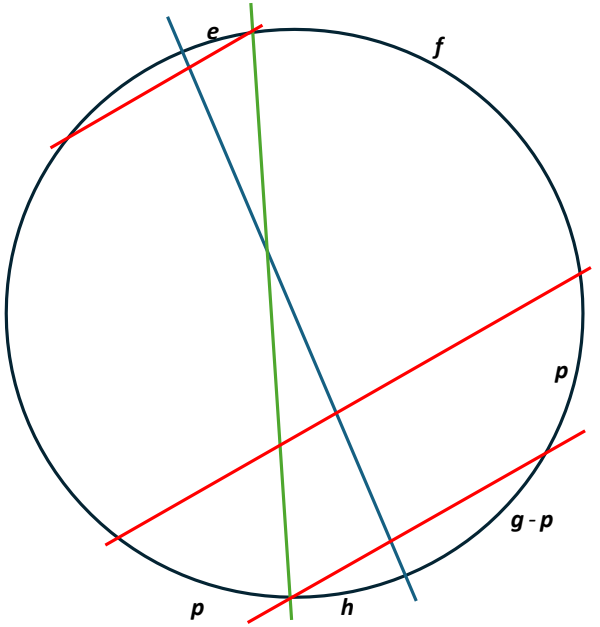
By the Interior Angle Theorem:

$a = e + h$

$b = f + p$

$c = g + q$





NOTES:

All a, b, c triangles images can be represented, up to a rotation or reflection, using the two constructions shown in the left column.
When n is even, two versions are possible, one with a vertex triangle (top row), the other with no vertex triangle (bottom row).
The first column shows the vertex setup for each version. The second column shows a typical interior apex based on lines parallel to the original lines using vertices of the n-gon.
The third column shows **red** parallel lines representing the base of the largest inverted triangle and largest upright triangle with apex at the internal apex.

The number of vertices between outer **red** lines is $f + p$, so apex count is $f + p + 1 = b + 1$.