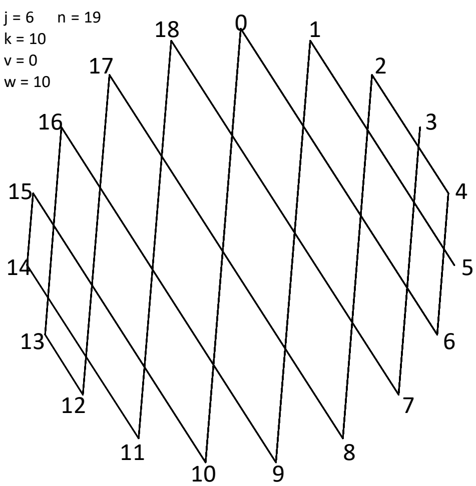
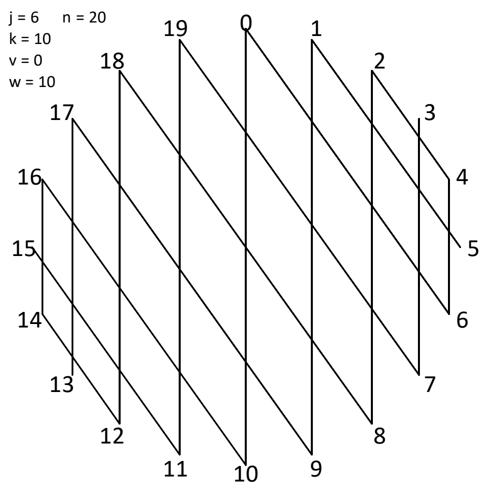
About Interior and Vertex Apexes in an *a,b,c* Image

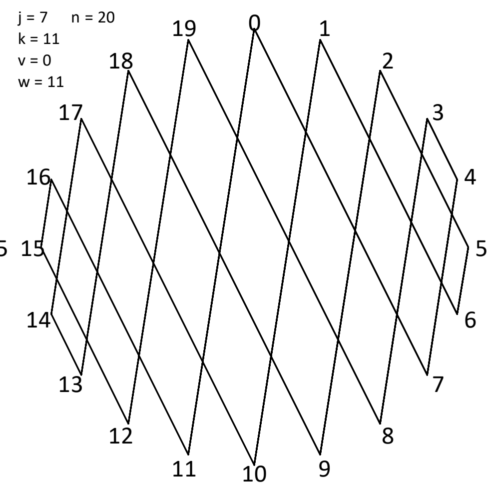
Interior Apexes. Our general strategy for counting triangles has been to count along the perimeter and count along arcs in the interior. In doing so we use the sharpest angle as the distinguished point of the triangle. This requires we measure what happens when lines in the ***b*** and ***c*** directions intersect. Lines in the ***a*** direction (opposite the sharpest angle) allow us to count bases, but for finding apexes, they are simply a distraction. By setting $V = 0$ and $W = K$ we eliminate lines in the ***a*** direction. Six examples are shown (**A**-**F**). The left and middle use the standard VT setup with $J = b$ and $K = a+b$ while the right uses the no VT setup of $J = b+1$, $K = a+b+1$. The top row has $a = 4$ and the bottom has $a = 5$ because you can see $3 = a-1$ arcs of internal apexes in the top row and 4 in the bottom. Labels and arc apex counts are noted beneath each image and are easy to obtain using the vertices of the ***K*** direction lines along the bottom. Consider **A**, for example. The left-most line in ***K*** direction in the upper arc uses (16,13) and right-most uses (3,7) so there are 7 = 13-7+1 lines in the upper arc. The second arc has the same left and right-most lines and the third is from (17,12) to (4-6) or 7.



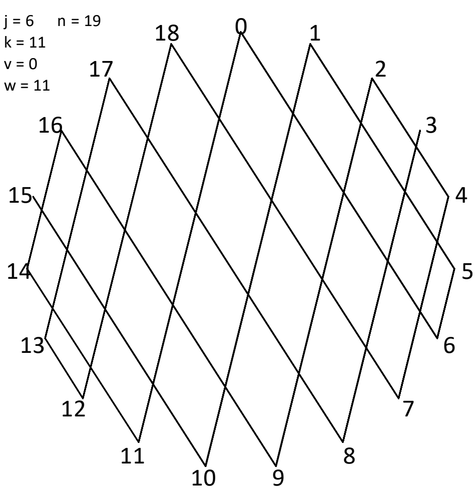
A. Interior = 7+7+7 = 21



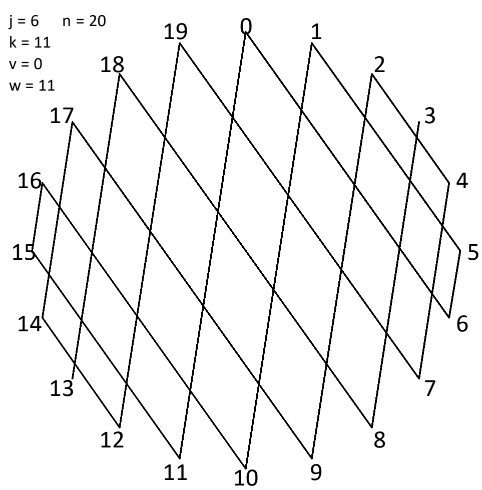
B. Interior = 8+7+8 = 23



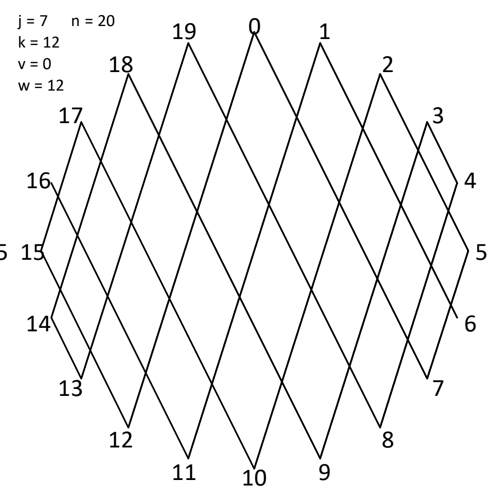
C. Interior = 7+8+7 = 22



D. Interior = 7+6+7+6 = 26



E. Interior = 7+7+7+7 = 28



F. Interior = 7+7+7+7 = 28

Odd *n*. If ***n*** is odd like **A** and **D**, there are $(n-3)/2$ lines with interior vertices (one side has a single vertex (e.g. 3) the other (e.g. (12,13)) spans one vertex). The rest have interior apexes. Interior apexes increase from *side to side* by 1,2, …, $a-1$ regardless of whether ***a*** is even or odd. The sum of these is Δ_{a-1} and the rest $(n-3)/2-(a-1)$ lines have $a-1$ apexes each. So,

Total internal apexes given odd ***n***	Total apexes = $a(a-1)/2+(a-1)((n-3)/2-(a-1))$,
which simplifies to	Total apexes = $(a-1)n/2-a^2/2+1/2$.

Even *n*. When ***n*** is even, the situation is a bit more complex. If the smallest line in both directions have an even number of vertices spanned (like **B** with single lines at vertices 3, 5, 13, and 15) the results are different from when both are odd (like **C**). When the two directions are mixed (like **E** and **F**), then the counting remains balanced.

When we are interested in counting triangles on apex arcs, our focus is on where the perimeter starts on the right and stops on the left as well as the starting values at those apexes. The start value at right is based on the b direction (like lines (2,4) in **B** and **E** and (3,4) in **C** and **F** for the upper arc) and the end value at left is based on the c direction (like lines (16,14) in **B**, (16,15) in **C** and **E**, and (17,15) in **F** for that same arc).

Here we count interior apexes from side to side like we did above with odd n rather than by arcs. The answer hinges on whether the smallest line spans one vertex or two and whether a is even or odd. Consider the lines in the J direction.

Even J. If J is even like **B** and **E**, then the smallest line spans two vertices on both sides. There are (n-2)/2 lines in this direction and all of them have internal apexes. The smallest have 1 (like on lines (2,4) and (12,14)) and they increase by 2s in size until there are a-1 apexes on each line.

Even a. If a is even, like **B** where a = 4, then each step up approaching a-1 is 2. In **B**, that occurs on the second 2 step because the count from side to side is: 1,3,3,3,3,3,3,3,1 on the 9 lines. Total apex count in **B** is 23 without using arcs.

Odd a. If a is odd, like **E** where a = 5, then the last step up approaching a-1 is 1. In **E**, that occurs after 2 steps because the count from side to side is: 1,3,4,4,4,4,4,3,1 on the 9 lines. Total apex count in **E** is 28 without using arcs.

In both instances, the steps leading up to (a-1) are the sum of odd numbers, which we know is a perfect square of the number of steps (in both instances $1+3=2^2$) by gnomons.

More generally, **B**. For a given even n, J, and a, Total apexes = $2(a/2)^2+(a-1)((n-2)/2-a)$.

E. For a given even n, J, and odd a, Total apexes = $2((a-1)/2)^2+(a-1)((n-2)/2-(a-1))$.

Odd J. If J is odd like **C** and **F**, then the smallest line spans one vertex on both sides. There are n/2 lines in this direction and all but 2 of them have internal apexes. The smallest with internal apexes have 2 (like on lines (2,5) and (12,15)) and they increase by 2s in size until there are a-1 apexes on each line.

Even a. If a is even, like **C** where a = 4, then each step up approaching a-1 is 2. In **C**, that occurs only on the first step because the count from side to side is: 2,3,3,3,3,3,3,2 on the 8 lines. Total apex count in **C** is 22 without using arcs.

Odd a. If a is odd, like **F** where a = 5, then the last step up approaching a-1 is 2. In **F**, that occurs after 2 steps because the count from side to side is: 2,4,4,4,4,4,4,2 on the 8 lines. Total apexes in **F** is 28 without using arcs.

In both instances, the steps leading up to (a-1) are the sum of even numbers, which we know twice the value of a triangular number of steps (in the first instance 1 and the second 2).

More generally, **C**. For a given even n, odd J, and even a, Total apexes = $a(a-2)/2+(a-1)((n/2-2-(a-2)))$.

F. For a given even n, odd J, and odd a, Total apexes = $(a+1)(a-1)/2+(a-1)((n/2-2-(a-1)))$.

Algebra aside: Note that **E** and **F** produce the same values, but **B** and **C** differ by 1. This can be shown more generally.

For a given even n, J, and odd a (like **E** of 28),

Total apexes
$= 2((a-1)/2)^2+(a-1)((n-2)/2-(a-1))$
$= (a-1)^2/2+(a-1)((n-2)/2-(a-1))$
$= (a-1)((a-1)/2+(n-2)/2-(a-1))$
$= (a-1)((n-2)/2-(a-1)/2)$
$= (a-1)(n/2-1-a/2+1/2)$
$= (a-1)(n/2-a/2-1/2)$ so, **E** and **F** produce the same value (and that equation works for odd n as well).

For a given even n, odd J, and odd a (like **F** of 28),

Total apexes
$= (a+1)(a-1)/2+(a-1)((n/2-2-(a-1)))$.
$= (a-1)((a+1)/2+n/2-2-a+1)$
$= (a-1)(a/2+1/2+n/2-2-a+1)$
$= (a-1)(a/2+n/2-a-1/2)$
$= (a-1)(n/2-a/2-1/2)$

For a given even n, J, and a (like **B** of 23),

Total apexes
$= 2(a/2)^2+(a-1)((n-2)/2-a)$
$= a^2/2+(a-1)(n/2-1-a)$
$= a^2/2+(a-1)n/2-(a-1)(a+1)$
$= a^2/2+(a-1)n/2-(a^2-1)$
$= (a-1)n/2-a^2/2+1$
$= (a-1)n/2-a^2/2+1$
$= (a-1)n/2-a^2/2+1$

For a given even n, odd J, and even a (like **C** of 22),

Total apexes
$= a(a-2)/2+(a-1)((n/2-2-(a-2)))$
$= a(a-2)/2+(a-1)(n/2-2)-(a-1)(a-2)$
$= (a-1)(n/2-2)+(a/2-a+1)(a-2)$
$= (a-1)n/2-2(a-1)+(1-a/2)(a-2)$
$= (a-1)n/2-2a+2-(a/2-1)(a-2)$
$= (a-1)n/2-2a+2-(a^2/2-2a+2)$
$= (a-1)n/2-a^2/2$ which differ by 1 like **B** v. **C**.

And $(a-1)(n/2-a/2-1/2) = (a-1)n/2-(a-1)(a+1)/2 = (a-1)n/2-(a^2-1)/2 = (a-1)n/2-a^2/2+1/2$ is sandwiched between **B** and **C**.

The total triangles count depends on the perimeter as well as interior. We just saw that there are circumstances where the VT and no VT style images produce different numbers of interior apexes. Table 1 focuses attention on interior apexes and Table 2 follows that up with an analysis of perimeter apex counts.

In general terms, there are two possibilities for ***a*** and ***b***, each can be either even or odd. Table 1 lays out these possible solutions and provides summary information on perimeter and first interior triangles counts for both even and odd ***n***. We know that for even ***n***, the same pattern occurs with 180° rotational symmetry but for odd ***n***, there is no such rotational symmetry. The first thing to notice is that we need not examine 8 figures for even ***n*** because even **a**, even ***b*** VT Figure **B** has the same attributes as even ***a***, odd ***b*** no VT. Similarly, **C** has the same attributes as even ***a***, odd ***b*** VT, and so forth.

Table 1		Vertex Triangle						No Vertex Triangle							Odd # of arcs
	Figure	Perimeter		1st Interior		# Interior Apexes		Figure	Perimeter		1st Interior		# Interior Apexes		
<u>Even *n*</u>		L	R	L	R	1st	2nd		L	R	L	R	1st	2nd	Y/N
even ***a*** even ***b***	**B**	2	2	2	2	$(n-a)/2$	$(n-a-2)/2$	**C**	1	1	3	3	$(n-a-2)/2$	$(n-a)/2$	Y
odd ***b***		1	1	3	3	$(n-a-2)/2$	$(n-a)/2$		2	2	2	2	$(n-a)/2$	$(n-a-2)/2$	Y
odd ***a*** even ***b***	**E**	1	2	3	2	$(n-a-1)/2$	$(n-a-1)/2$	**F**	2	1	2	3	$(n-a-1)/2$	$(n-a-1)/2$	N
odd ***b***		2	1	2	3	$(n-a-1)/2$	$(n-a-1)/2$		1	2	3	2	$(n-a-1)/2$	$(n-a-1)/2$	N
<u>Odd *n* (VT)</u>	<u>Top Arc</u>							<u>Bottom Arc</u>							
even ***a*** even ***b***	**A**	1	2	3	2	$(n-a-1)/2$	$(n-a-1)/2$		1	2	3	2	$(n-a-1)/2$	$(n-a-1)/2$	Y
odd ***b***		2	1	2	3	$(n-a-1)/2$	$(n-a-1)/2$		2	1	2	3	$(n-a-1)/2$	$(n-a-1)/2$	Y
odd ***a*** even ***b***	**D**	2	2	2	2	$(n-a)/2$	$(n-a-2)/2$		1	1	3	3	$(n-a-2)/2$	$(n-a)/2$	N
odd ***b***		1	1	3	3	$(n-a-2)/2$	$(n-a)/2$		2	2	2	2	$(n-a)/2$	$(n-a-2)/2$	N

When the smallest perimeter value is 1, like at left and right with **C**, at left with **E**, at left with **A**, at right with **F**, or at bottom with **D**, the first interior triangles count is 3 but if the smallest perimeter value is 2, like at left and right with **B**, at right with **E**, at right with **A**, at left with **F**, or at top with **D**, the first interior triangles count is 2. From there, the triangles count increase by 2 up to ***b*** on perimeter vertices and to ***b***+1 on interior apex arcs. Once these values occur from the left and right, plateau values of ***b*** and ***b***+1 are maintained on the perimeter and on interior arcs respectively.

Focus on the lead up to *b* or *b*+1. When right and left have different values on the perimeter or on an interior arc, like figures **A**, **E**, and **F**, one can jump from side to side to see the lead up to the plateau value as $\triangle_b$ for the perimeter and as $\triangle_{b+1}$-1 for an interior arc.

On the other hand, with **D** one can jump from bottom to top on each side (1, 2, 3, …, ***b***) to obtain $\triangle_b$ for the perimeter and one can obtain $\triangle_{b+1}$-1 by counting two adjacent interior arcs (2, 3, …, b+1) since interior apex arcs alternate between starting at 2 and 3. Notice that there are an even number of interior arcs if ***a*** is odd so they can be thought of in pairs.

The strategy employed for **D** does not work with **B** and **C** since there are the same values at top and bottom along the perimeter and there are an odd number of internal apex arcs since ***a*** is even. As a result, even ***n*** with even ***a*** and ***b*** VT and no VT images may produce different interior total triangles counts, even in the absence of concurrence issues.

Perimeter Apexes. The total number of triangles in an image is a function of both the number of perimeter vertices that have non-zero apex counts as well as the number of interior apexes. All interior apexes have triangle counts from 2 to ***b***+1 and perimeter apex counts are bounded by 1 and ***b*** but not all vertices have positive apex counts. Just like interior apexes, the number of vertices with positive apex counts depends on whether ***n***, ***a*** and ***b*** are even or odd, and whether the image is VT or no VT (if ***n*** is even). These patterns are summarized using four images, **G**, **H**, **L** and **M** and in Table 2.

Images G, H, L, and M and Table 2. The four images highlight smallest lines in the two directions. The points of the two smallest apex angles are at x and y. Vertex x is the smallest numbered vertex with an apex count greater than zero, counting in the clockwise direction starting at 0 and y is the first vertex with an apex count greater than zero after x. For each vertex between x and y, apex counts are zero. These four images allow us to determine how many zero vertices are between x and y.

G and **H** show the two possibilities when ***a*** is even and **L** and **M** show the two possibilities when ***a*** is odd. (Although specific values for ***a*** are shown (6, and 7), the same stylistic patterns emerge for any even or odd ***a***.) The smallest lines can span either 1 or 2 vertices and hence have an apex count of either 1 or 2. The smallest line starting at x spans a single vertex if ***n*** is odd and ***b*** is odd. If ***n*** is even, x spans a single vertex if ***J*** is odd meaning either ***b*** is odd and the image is VT style, or ***b*** is even and the image is no VT style (since ***J*** = ***b***+1 for no VT style images). The smallest line at x spans 2 vertices when the opposite occurs. The size of the smallest line at y depends on whether ***a*** is even or odd as well as the apex count at x. The details are explained in Table 2 which also notes the links to the initial six images **A**-**F**. Note that even ***n*** images have rotational symmetry so x and y and x+***n***/2 and y+***n***/2 values are the same in each case. Since odd ***n*** does not exhibit rotational symmetry, the left-hand-side smallest vertices are provided for each of the four (x, y) possibilities.

In short: The standard VT setup sets ***J*** = ***b*** and ***K*** = ***b***+***a*** so that the smallest line in the ***b*** direction is 2 if ***b*** is even and 1 if ***b*** is odd. The smallest line at x in the no VT setup is the reverse since ***b***+1 is odd if ***b*** is even (hence the smallest line at x is 1) and even if ***b*** is odd (hence the smallest line at x is 2).

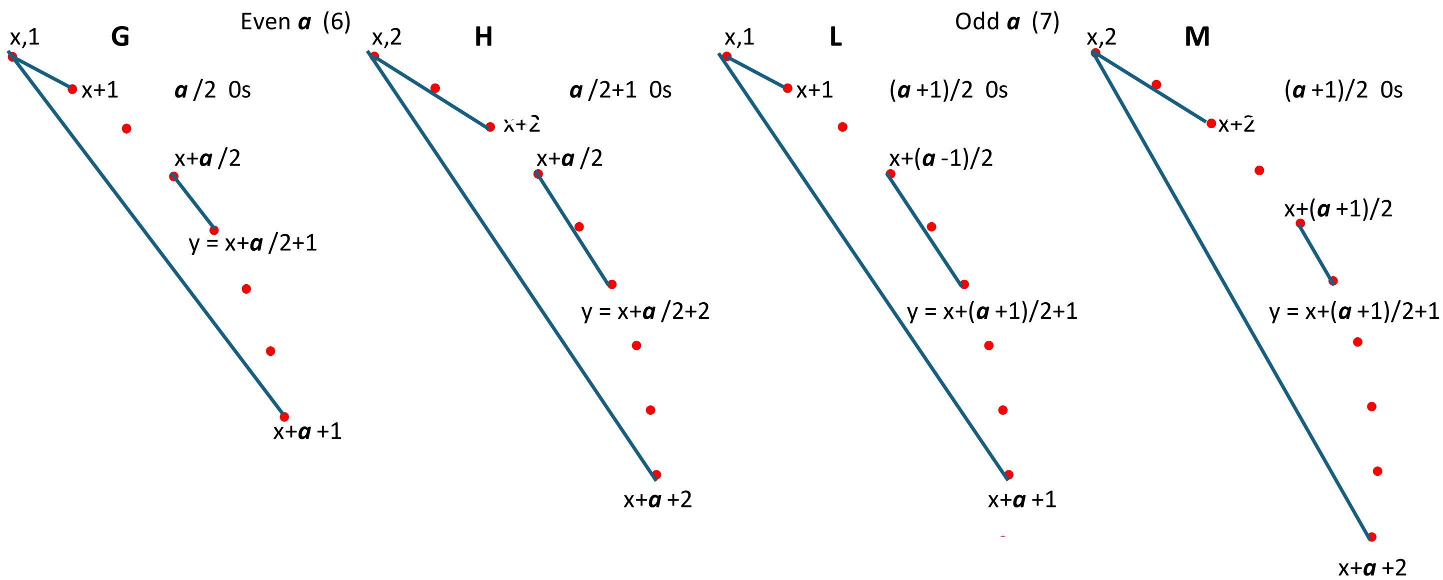


Table 2

	R Arc*		Figures	***n*** and ***b*** given ***a***, x, and y — Even ***n***: # 0s between x and y	***b***	VT?	***b***	VT?	# Vertices Apex > 0	***b*** given VT at vertices (0,***b***,***a***+***b***)	Odd ***n***: Apex at (***n***-1)/2+ x	y	x+1	y+1	# 0s between smallest values on L	# Vertices Apex > 0
	x	y														
even ***a***	1	1	**C, G**	***a***/2	Even	nVT	Odd	VT	***n***-***a***	Odd	2	0	0	2	***a***/2+1	***n***-***a***-1
	2	2	**A, B, H**	***a***/2+1	Even	VT	Odd	nVT	***n***-***a***-2	Even	3	1	1	3	***a***/2	***n***-***a***-1
odd ***a***	1	2	**F, L**	(***a***+1)/2	Even	VT	Odd	nVT	***n***-***a***-1	Odd	2	1	0	3	(***a***+1)/2	***n***-***a***-1
	2	1	**D, E, M**	(***a***+1)/2	Even	nVT	Odd	VT	***n***-***a***-1	Even	3	0	1	2	(***a***+1)/2	***n***-***a***-1

*The x vertex number depends on even or odd ***b*** and VT or no VT style. y's vertex is y = x + # 0s + 1.

x	style	***b***	Vertex	x	style	***b***	Vertex
1	VT	Odd	(***b***-1)/2	1	nVT	Even	***b***/2
2	nVT	Odd	(***b***-1)/2	2	VT	Even	***b***/2-1

*x and y from **G**, **H**, **L**, **M**. The decline in apex count from vertex 0 is 2 per vertex for VT style but 1 at vertex 1 then 2 thereafter for nVT style. Even ***n*** have 180° rotational symmetry so number of zero vertices on L is same as R. Odd ***n*** images potion of the table uses VT style.

Summary of number of apexes with non-zero apex counts. A comparison among VT and no VT scenarios suggests that the total number of apexes with non-zero apex counts depends on whether ***a*** is even or odd given an even ***n***. From Table 1 and Figures **B** and **C** we see that if ***a*** and ***b*** are even, the total number of interior apexes is one larger in VT style than no VT style. By contrast, Table 2 shows that the total number of non-zero perimeter apexes is 2 larger in one style than the other for even ***a*** but which is larger depends on whether ***b*** is even or odd. Given these differences, no single counting rule will emerge for even ***n***, even in the absence of concurrence issues.