Odd Scalene Triangles

We put together material from the last couple of sections to provide a formula for the number of triangles in an odd scalene triangles image. The only requirement is that ***a*** < ***b*** < ***c*** and ***a***+***b***+***c*** = ***n*** is odd. Our analysis focuses on three parameters, ***a***, ***b***, and ***n*** with ***a*** < ***b***, ***n*** ≥ ***a***+2***b***+1, and ***n*** odd.

Six images. The attributes of odd triangles images allow us to streamline the number of images we need to examine to obtain a unified formula. We could discern the formula by comparing any two odd ***n*** images, ***n*** and ***n***+2 because the attributes of odd ***n*** images discussed in Tables 1 and 2 two sections ago show that we can employ the formula for triangular numbers to deal with the up and down parts of both perimeter and interior apex counts. The only question is the size of the interior and perimeter plateaus. Rather than provide a single comparison, we initially provide two, and we provide the intermediate even ***n*** values to show why we separately analyze even and odd values of ***n***.

The six images share the three attributes in the lower right corner of images in the last section: each has number of concurrences in the image in yellow; the triangles count in red; and a label beneath, starting at **XI** so that we can readily make comparisons between these two sections. The top and bottom rows are odd ***n*** and the middle row has even ***n***. Both top values were chosen with ***c*** = ***b***+1 so they are as close to isosceles as possible while being scalene. Finally, both were chosen with ***a*** = 4 so that there are an odd number of interior arcs.

If you quickly scan the left column and the right column you should see the right "half" of all three images, shown in **red**, have the same values at vertices and interior values. This is because each column is based on the same standard form VT values for ***J***, ***K***, ***V***, and ***W***; the only thing that varies is ***n***. When ***n*** is one larger (the even numbered middle row) the rest changes. These are the numbers shown in **green** and **black** in the middle row.

What happens when *n* changes by 1? The two columns show why problems arise if you try to create a unified formula for scalene triangles. As we saw in the last section, if there are concurrences, they interrupt the formula progression. Both even ***n*** images have sporadic concurrences, **XII** has one and **XV** has two, and each sporadic concurrence comes in pairs due to 180° rotational symmetry inherent in even ***n*** images so there are 2 concurrences in **XII** and 4 in **XV**. The increase in total triangles count (plus concurrences) from **XI** to **XII** to **XIII** is 14, 14 but the increase from **XIV** to **XV** to **XVI** is 17, 16. The even ***a***, odd ***b*** left hand side produces regular changes but the even ***a***, even ***b*** right hand side does not.

What happens when *n* changes by 2? When ***n*** is two larger (the next larger odd numbered bottom row) the right AND left half looks like the top row; the only difference is an extra ***b*** at top and bottom and an extra ***b*+1** on each interior apex arc (these are the values shown in **purple**). The total triangles apex count increases by 2***b*** + (***a***-1)(***b***+1). These are the plateau values on the perimeter and on interior arcs, respectively. To count the total triangles in an image, it helps to consider the perimeter and interior apexes separately.

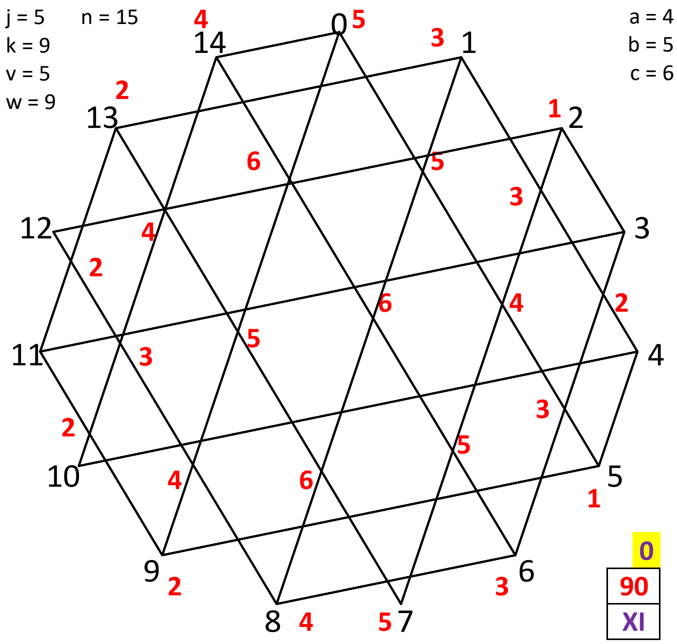
Perimeter: Both smallest odd ***n*** examples shown in **XI** and **XIV** exhibit a similar perimeter pattern: the top and bottom perimeter values exhibit an asymmetry from side to side, stylistically similar to images **A**, **G** and **H**, rows 5 and 6 from Table 1, and rows 1 and 2 of Table 2 two sections earlier. (We examine when the asymmetry is from top to bottom in image **XVII**.) By switching between left and right all values from 1 to ***b*** are present, the sum of which is Δ_{b} = ***b***(***b***+1)/1. Notice that the upper row has a single value of ***b*** at top and bottom and the bottom row has two at top and bottom. Put another way, the plateau value ***b*** increases by 1 at top and bottom for each increase in ***n*** of 2, starting at the smallest value for **c** which was chosen as ***c*** = ***b***+1. The perimeter triangles apex count is thus,

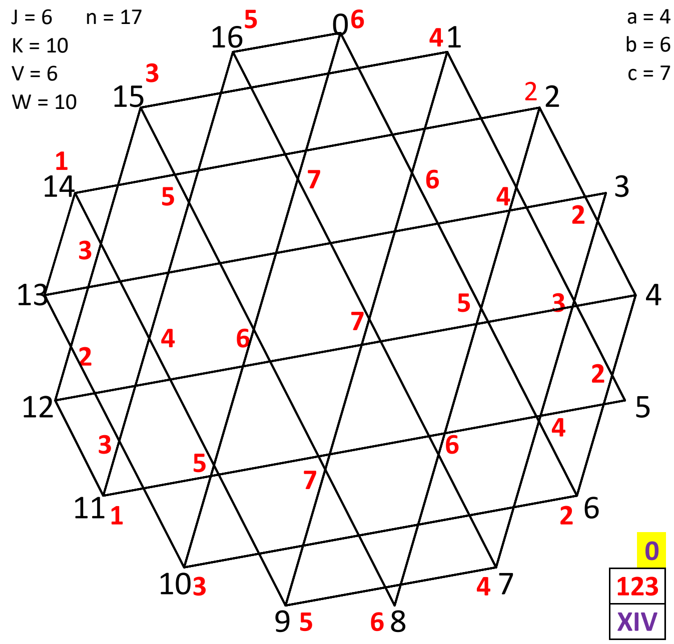
Total perimeter triangles count = ***b***(***b***+1) + ***b***(***n***-***a***-2***b***-1).

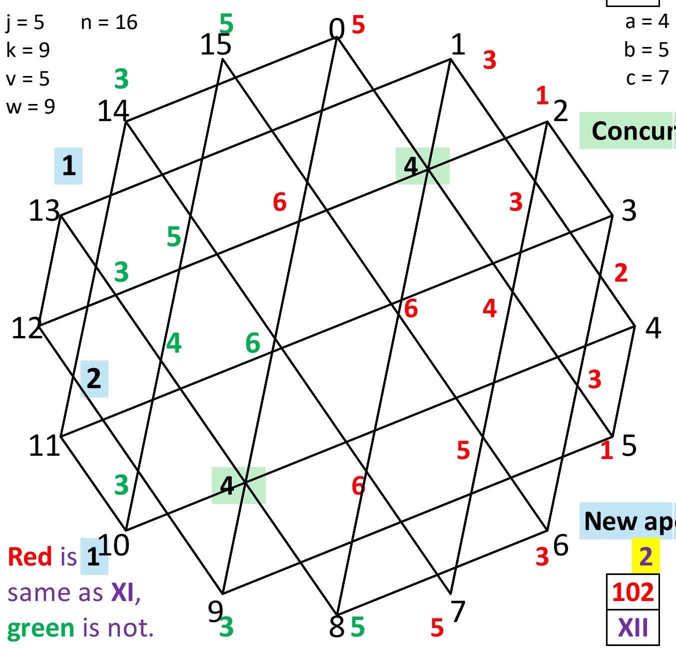
Interior Apexes: When perimeter counts are asymmetric from side to side, so are interior arcs. They increase from 2 to ***b***+1 which sums to Δ_{b+1}-1 before reaching this plateau value. When ***c*** = ***b***+1 this value is only achieved once on each arc but as ***n*** increases by 2, the number of ***b***+1s increases by one (there are two 6s per arc in **XIII** and two 7s per arc in **XVI**). The total of interior apex triangles counts on each of the ***a***-1 arcs is the sum of the up/down and plateau values, so,

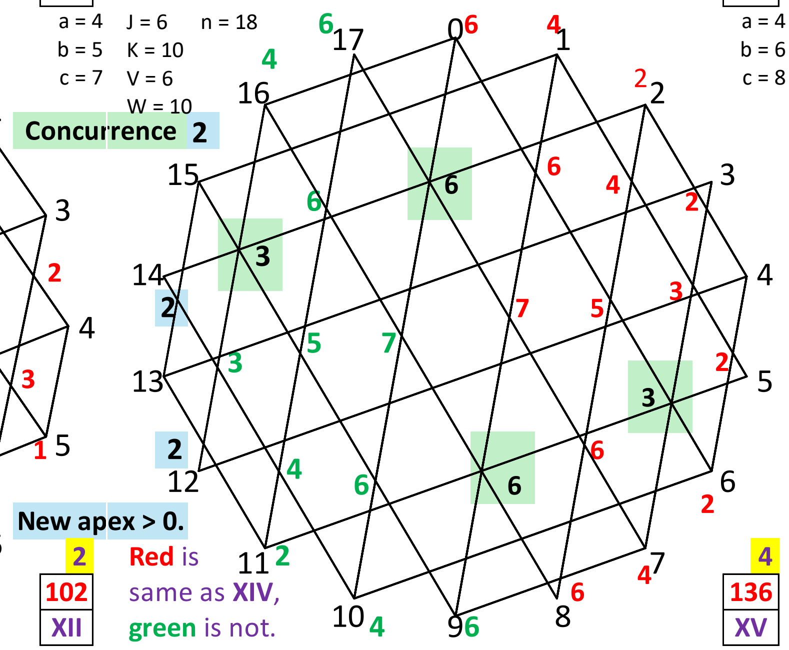
Total interior triangles count = (***a***-1)([(***b***+1)(***b***+2)/2-1] + [(***b***+1)(***n***-***a***-2***b***-1)/2]).

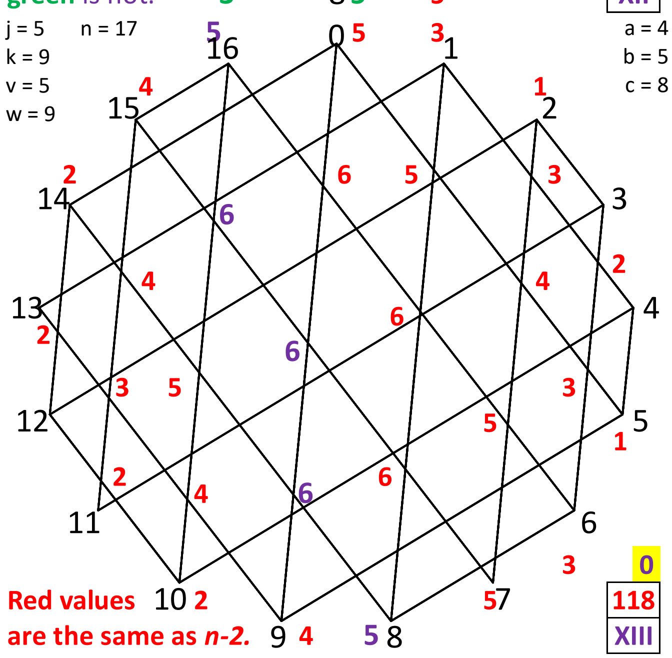
In sum: The total triangles count for any odd scalene triangle is the sum of these two pieces.

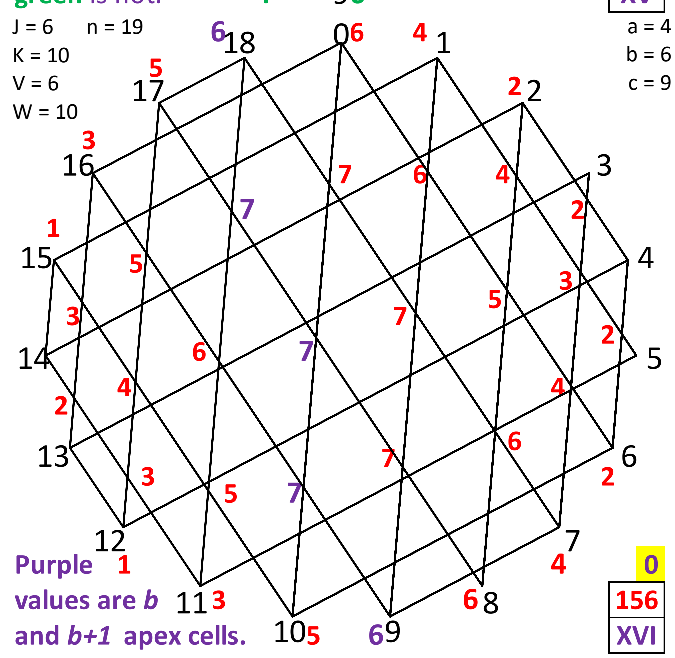

j = 5
k = 9
v = 5
w = 9
n = 15
a = 4
b = 5
c = 6
0
90
XI


J = 6
K = 10
V = 6
W = 10
n = 17
a = 4
b = 6
c = 7
0
123
XIV


j = 5
k = 9
v = 5
w = 9
n = 16
a = 4
b = 5
c = 7
Red is 1 same as XI,
green is not.
2
102
XII


J = 6
K = 10
V = 6
W = 10
n = 18
a = 4
b = 6
c = 8
Concurrence 2
New apex > 0.
Red is
same as XIV,
green is not.
4
136
XV


j = 5
k = 9
v = 5
w = 9
n = 17
a = 4
b = 5
c = 8
Red values
are the same as n-2.
0
118
XIII


J = 6
K = 10
V = 6
W = 10
n = 19
a = 4
b = 6
c = 9
Purple
values are b
and b+1 apex cells.
0
156
XVI

About even and odd angles. The formula for odd scalene triangles is most easily explained by starting with the largest angle ***c*** one larger than the middle angle ***b*** as was done with **XI** and **XIV**. This means that if one angle (of ***b*** and ***c***) is even, the other is odd. Since the sum of the three angles is odd, the third angle must be even, which is why ***a*** = 4 was chosen for both examples.

One property of whole numbers is that if the sum of numbers is odd, then there must be an odd number of odd numbers in the sum. In the current context that means that if ***a*** is odd, then then either both ***b*** and ***c*** are odd, or both are even. In both instances, the smallest difference between ***b*** and ***c*** is 2, not 1.

Odd *n* and *a*. When ***a*** is odd, there are an even number of internal apex arcs ***a***-1. Choosing ***a*** = 5 provides enough arcs so that the internal pattern becomes clear. Two choices for ***b*** and ***c*** are both even or both odd. Given ***b*** > ***a***, the smallest ***n*** version that works in this instance is ***b*** = 6 and ***c*** = 8 so ***n*** = 19. This is shown as **XVII** but it is the same as image **D** (with lines in the ***a*** direction suppressed), **M**, and row 4 of Table 2 from two sections earlier. To aid counting this image has triangles counts in three colors, **red**, **blue**, and **black**. Instead of counting from side to side, count by color.

Perimeter: The top up and down values are even and the bottom are odd. The same point applies as above; all numbers from 1 (at vertices 6 and 12) to ***b*** (here ***b*** = 6 at vertices 0 and 18) are obtained if we count zig-zag top and bottom rather than from side to side. The right side is shown in **red** and the left is shown in **blue**. As above, there are $2\Delta_b$ total triangles with apexes at these twelve (2***b***) vertices.

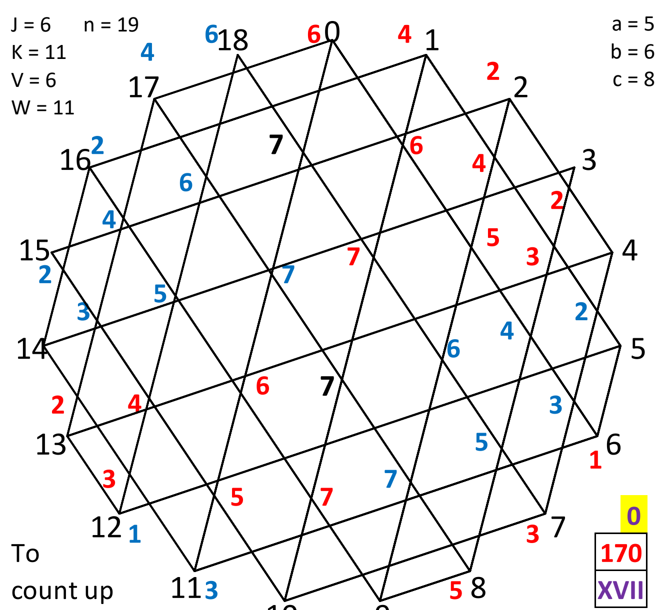
Interior Apexes: Just like above, there are ***a***-1 (here 4) interior apex triangles counts in the up and down portion of interior apexes taking on the values from 2 to ***b***+1 (here 7). But here apex arcs have either even or odd up and down counts not both. Notice that since ***a*** is odd, ***a***-1 is even so that these values can be counted zig-zag in pairs. The upper right is in **red** starting just below vertex 3, lower right is in **blue** starting just above vertex 5, upper left is in **blue** starting just below vertex 15, and lower left is in **red** starting just above vertex 13. As above, there are $(a-1)[\Delta_{b+1}-1]$ total triangles counted in the **red** and **blue** interior apexes.

Apex counts in **black***:* The three remaining triangles counts are in **black**: there are two **7**s (at middle of the first and third interior arcs); and one **6** (at vertex 9). These values are plateau counts that occur because ***c*** = ***b***+2. Specifically, the 6 is from the perimeter part of the equation: ***b***(***n***-***a***-2***b***-1) = ***b***; and the two 7s are from the interior apexes part of the equation: (***a***-1)[(***b***+1)(***n***-***a***-2***b***-1)/2] = (***a***-1)(***b***+1)/2 = 2(***b***+1) given ***a*** = 5, ***b*** = 6, and ***n*** = ***a***+***b***+(***b***+2).

In sum: The total triangles count for odd ***n*** and odd ***a*** follows the same formula obtained above for odd ***n*** and even ***a***.

The only difference if we had used ***b*** = 7 and ***c*** = 9 is that the ***n*** = 21 image is stylistically like **L** and row 3 of Table 2 from two sections earlier so that the odd values are on top and the first interior arc rather than even like **XVII**.

A final note. Image **XV** is the ***n*** = 18 version of the ***n*** = 20 image **V** from the previous section. By comparing these two images, one can readily obtain a general formula for even ***a***, ***b***, ***n*** VT images. This is left as a Challenge Question.



To count up and down, focus on the 5 red 6 and blue separately.