Vertex Triangles Take 2: Why There are Two Image Types for Even n, but One for Odd n

One of the more interesting patterns observed in prior chapters was that when n is odd there is only one image type but there are two when n is even. As we explore this issue, we will find that one version must include two vertex triangles, and the other one does not have any vertex triangles. The reason has to do with line lengths in various directions.

Constructing an Image with a Vertex Triangle. Any triangles image is based on three angles, a, b, and c, with $a \le b \le c$ and $a+b+c = n$. If we set $j = v = b$ and $k = w = a+b$ and $n = a+b+c$ in the *General Triangles* model, the resulting image has a vertex triangle at 0-j-k. This is the triangles set-up shown in the top left image in the *On Maximum Apex Counts* section, reproduced at top left below.

The middle-sized leg, **0*j* in blue** is downward sloping, the longer-sized leg, 0k is steeply upward sloping if $k > n/2$ or more steeply downward sloping than 0j if $k < n/2$ (like shown in **green** below), and $jk = vw$, the short side shown in **red**, is shallowly upward sloping (unless $a+b < n/4$ (very steep negative slope), or $b = c$ (horizontal)). By construction, $k-j = a$.

"One if by Odd, Two if by Even." If there is a second vertex triangle in the image, it would have to have another line in the jk direction that spans a vertices. This is not possible if n is odd because odd n have one line each of length 1 to $(n-1)/2$, or $(n-1)/2$ lines in each direction according to P2.2. Thus, there is only one vertex triangle in an odd n triangles image.

However, if n is even there is 180° rotational symmetry and hence there is a vertex triangle with peak at $n/2$ and spanning vertices $n/2$ to $n/2+b$ to MOD($n/2+b+a$, n) where the third vertex must be shown in modular form because it may be the case that $n/2+b+a > n-1$. This symmetry is confirmed by the fact that with even n, there are $n/2$ lines in a direction if the spans are odd, but **n**/2-1 lines in a direction if the spans are even (since, in that instance, there is a vertex between the smallest span of 2 vertices on each side).

Constructing an Image with No Vertex Triangle. The bottom left image is obtained by setting $j = b+1$, $k = w = a+b+1$, and $v = b$. When n is even, this produces an image with no vertex triangles because there is no line in the vw direction that spans a vertices. The closest is a line that spans $a+1$ (shown in **red**) or $a-1$ because closest parallel lines differ by two vertices by construction. This is why parallel lines in a direction are either even or odd but not both for even n. The triangle shown is the closest to an a, b, c -angled vertex triangle in the image.

When n is odd, there is a line that spans a vertices in the vw direction, it is on the "other" side of the image from the a+1 spanning line shown in **red**. The apex of the vertex triangle is just past the bottom of the odd n-gon at vertex $(n+1)/2$. Add b vertices to find the next point, or vertex $(n+1)/2+b$ which is a line that spans b vertices in the **0*j*** direction. The third vertex spans a vertices in the **vw** direction and is MOD($(n+1)/2+b+a$, n). Two examples with fixed a and b (of $a = 3$ and $b = 4$) are shown at middle and right. For $n = 13$, the vertex triangle is 7-11-1 and for $n = 23$, the vertex triangle is 12-16-19.