Concurrence in Triangles Images and Diagonals of Regular Polygons

In the last chapter we argued that for counting purposes, it is worthwhile to ignore concurrences which disrupt counting patterns. Those disruptions happen when the concurrences are not a regular part of the image. In certain instances, they are a regular part of the image rather than a disruption. We examine both types of concurrence in this chapter.

We start with a brief introduction to concurrence, followed by an examination of concurrences that occur when $\boldsymbol{n} = 12$ since it is useful to examine topics in the context of a familiar regular polygon, which is why we often resort to a clockface. We follow this up with an examination of patterns in concurrences that occur by examining all triangles images from $\boldsymbol{n} = 3$ to $\boldsymbol{n} = 24$. This requires us to systematically think about how many distinct triangles images are possible for a given $\boldsymbol{n}$, which, as it turns out, is not a trivial question to answer.

These patterns lead us to examine patterns that occur in images using all lines from vertex to vertex rather than the subset of lines that are available in three directions. There are two types of lines from vertex to vertex, those that are adjacent to one another and those that are not. Lines connecting adjacent vertices form the edges or outer boundary of the polygon (also called sides) and those connecting non-adjacent vertices are **diagonals** that are interior to the polygon.

The total number of lines in this instance is $\boldsymbol{n}\cdot(\boldsymbol{n}-1)/2$ lines since each vertex is connected to all other $\boldsymbol{n}$-1 vertices by a line, but if you count this $\boldsymbol{n}$ times, you have counted all lines twice given that a line starts at one vertex and ends at another. Note that this is the $\boldsymbol{n}\text{-}1^{st}$ triangular number and it is also the formula for $\boldsymbol{n}$ choose 2. $\boldsymbol{n}$ of these lines are edges and the other $\boldsymbol{n}\cdot(\boldsymbol{n}-3)/2$ are diagonals. This compares to the much smaller number of lines if the lines are restricted to parallel lines in three directions. In that instance, the number of lines ranged between $3\boldsymbol{n}/2-3$ and $3\boldsymbol{n}/2$ when $\boldsymbol{n}$ is even and equals $3(\boldsymbol{n}-1)/2$ when $\boldsymbol{n}$ is odd (P2.2).

Concurrences in regular polygons leads us to examine further specific values of $\boldsymbol{n}$ for triangles images. We conclude by examining near concurrences and smallest triangles that result.

Regular concurrence. There are two types of regular concurrence, central concurrence and diameter concurrence, although at times diameter concurrences include central concurrence. We start with central concurrence.

Central concurrence. A central concurrence requires lines in all three directions to meet at the center. Put another way, it requires that all three are diameter lines of length $\boldsymbol{n}/2$. This can only occur when $\boldsymbol{n}$ is even, but not all even $\boldsymbol{n}$ triangles images have a central diameter. An image has an even $\boldsymbol{n}$ if all three angles are even, or if there are two odd and one even angle due to the properties of addition of whole numbers. But one cannot have three diameters if angles are both even and odd because all other lines maintain the same even- or odd-ness for the VT style but reverse even- and odd-ness in the no VT style of triangles images. The only way to obtain an all even or an all odd set of lines is to have $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ all even. Only half of all even angles images will have a central concurrence. If $\boldsymbol{n}$ is divisible by 4, $\boldsymbol{n}/2$ is even so any all even angles image will have a central concurrence if it is VT style and NO all even angles images will have a central concurrence if it is no VT style. On the other hand, if an all even angles image has $\boldsymbol{n}$ divisible by 2 but not 4, $\boldsymbol{n}/2$ is odd so any all even angles image will have a central concurrence if it is no VT style but not if it is VT style.

Diameter concurrence. A second form of systematic concurrence, diameter concurrence, includes a diameter line and two other lines of the same size intersecting on the diameter line. These lines are created in isosceles triangles images. The total number of concurrences on the diameter is $\boldsymbol{b}$-1. Diameter concurrences occur for VT style images if $\boldsymbol{n}$ is divisible by 4 and in no VT style images if $\boldsymbol{n}$ is divisible by 2 but not 4 since one of the lines must be a diameter.

3 Diameter concurrence. It is worth pointing out that even $\boldsymbol{n}$ equilateral triangles images have 3 diameter concurrences. This is not surprising when you note that equilateral triangles can be thought of as "super" isosceles triangles. An even more interesting point that we will discover in this chapter is that odd $\boldsymbol{n}$ equilateral triangles images (such as 3,3,3 or 5,5,5) have NO points of concurrence.

Other types of concurrence exist, but they are more sporadic in nature. We will explore each of these kinds of concurrence in greater detail in the sections to come. You can read about the mechanics of testing for concurrence in the mathematical appendix.