Counting Triangles Acknowledging Systematic Concurrence

We are now able to incorporate our understanding of triangles patterns with concurrence patterns. At the end of the last chapter we were able to provide a complete solution for odd n triangles images. We were able to do so because odd n images have only one style (all odd images have a single VT) and they have no concurrences as we learned in section 7.3.

The goal here is not to provide explicit equations for each possible alternative. Rather, the goal is to recognize when systematic concurrences alter the triangles count. When deciding how to describe the patterns of concurrence for even n images it makes sense to link n and $n+4$ rather than $n+2$ because the images switch between VT and no VT regarding when systematic concurrences appear. The easiest way to see this is to consider when central concurrences occur.

Central Concurrence. As noted at the start of this chapter, central concurrence requires lines in all three directions have diameter lines having length $n/2$. We know that all three directions must be even because all three angles must share the same even- or odd-ness and if all three angles are odd then $a+b+c = n$ is odd, and there is no vertex-to-vertex line of length $n/2$ for odd n. Any even n must be of the form $n = 4k$ or $n = 4k+2$.

When $n = 4k$, $n/2 = 2k$ is even. This means that $a,b,n-a-b$ VT style images will have a central concurrance and, as a result, one fewer triangle will be in the triangles image due to this central concurrence. But note that when n is 2 larger, the $a,b,n+2-a-b$ VT image will NOT have a central concurrence requiring a reduction of one as a result because now the diameter, $(n+2)/2 = 2k+1$, is odd. But when n is 4 larger, VT has a central concurrence because $(n+4)/2 = 2k+2$ is even.

Given a and b even, and $n = 4k+2$, $n/2 = 2k+1$ is odd. The VT and no VT images reverse even- and odd-ness in each direction so the no VT image will have a central concurrence (and one fewer triangle) since all three lengths are odd. As just noted above, 2 larger n produces an image with central concurrence for the VT but not the no VT image. But 4 larger n, $n = 4k+6$, once again produces an $n/2$ which is odd and hence has a central concurrence in the no VT image.

Diameter Concurrence. A diameter concurrence occurs when one of the lines is a diameter (hence n must be even) and all intersections on this line are concurrences. This only occurs if the diameter line is a line of symmetry. This requires the image to be an isosceles triangles image with the diameter line opposite the vertex angle which acts as the base line for some of the triangles.

If an isosceles triangle is not also equilateral, then either the smallest angle a, or the largest angle c, is the vertex angle. (In mathematical terms, the *vertex angle* is the angle formed by the two equal-sized sides (which are called the *legs* of the isosceles triangle), vertex angles need not be at a vertex of the n-gon. The *base angles* are equal and are opposite the legs, and the *base* is the side opposite the vertex angle.) The triangles image must be one of two forms: a,b,b or b,b,c where a and c are the vertex angles, respectively. Note that in both instances the base angle is b.

Given these two functional forms and an even n, we know that the vertex angle is even because $2b+a = n$ or $2b+c = n$. Unlike the odd isosceles triangles images discussed at the end of the last chapter, there are two styles, VT and no VT, with even isosceles triangles.

A Standard Form for Isosceles Triangles Images. When working with isosceles triangles images it is worthwhile to use the vertex angle as the distinguished point even if it is not the smallest angle because there are two locations for the smallest angle $a = b$ when c is the vertex angle. As a result, it makes sense to create an isosceles form for VT and no VT images. This form produces the same images as the Standard Form posed in section 6.7 if the vertex angle is smaller than the base angle with the base being horizontal for VT and slightly downward sloping for no VT. But if the vertex angle is larger than the base angle the base remains horizontal for VT and slightly downward sloping for no VT. The standard form is even easier than posed in section 6.7 since it is based on 2 values rather than 3 and one of those values is 0 for VT or 1 for no VT. These can be placed in the unprotected green cells and referenced in the yellow ***JKVW*** cells using the equations: J = **leg** + **no VT**, $K = n$ – **leg** + **no VT**, V = **leg**, $W = K$.

6 leg
0 no VT

These equations create images with horizontal and vertical symmetry if **no VT** = 0 and symmetry about the diameter connecting 0.5 and $n/2+0.5$ and the perpendicular bisector diameter to this line when **no VT** = 1. *Note that there are* ***vertex angle -1*** *internal arcs of vertex angles (rather than* a*-1 apex arcs) regardless of whether this angle is* a *or* c.

The 16 images on the next page use these equations to show the patterns that emerge. There are 8 pairs of images, the left of each pair is the horizontal VT version and the right is the slanted no VT version. The first 4 pairs show even values of ***n*** starting at 12 with **leg** = 5 and the last 4 show even values of ***n*** starting at 16 with **leg** = 6. The left set of pairs are ***n*** = 4***k*** and the right set are ***n*** = 4***k***+2. Comparing across images horizontally and vertically makes clear that ***n*** and ***n***+4 are more closely linked than are ***n*** and ***n***+2. Each image notes concurrence total and type in **blue** and prior sitings in **green**.

Diameter concurrence patterns. We know that the base line (opposite the vertex angle) is the diameter line and the vertex angle is even. As a result, VT images have horizontal diameter concurrences when ***n*** = 4***k*** (left column) and no VT images have diameter concurrences when ***n*** = 4***k***+2 (right column). There are ***b***-1 (**leg**-1) concurrent points on the diameter (4 for the first two rows in two off-center pairs, 5 for the last two rows in two off-center pairs plus a center).

What changes? As ***n*** increases for fixed ***b***, the vertex angle increases, and hence the number of vertex angle arcs increases. If one counts triangles via the vertex angle, the increase in total triangles as ***n*** increases comes from internal arcs increases (here vertex) not the count from the ***n*** vertices of the ***n***-gon.

Two other points. All but one of the non-diameter images have no points of concurrence but one, **leg** = 5, ***n*** = 18, has **8** sporadic (Sp) concurrences at 2 distances from the center since isosceles Sp concurrences come in 4s due to two perpendicular lines of symmetry. And the equilateral triangles image has **13** = 3·5-2 or **13** = 2·6+1. The first calculation notes that you cannot count the center 3 times and the second counts the two regular interior hexagons plus the center.