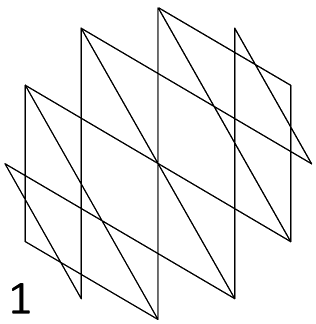
Concurrences Occurring Inside Dodecagon Triangles Images

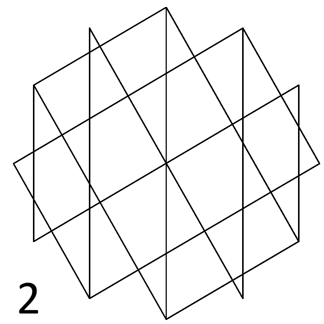
Generally, internal concurrences on triangles images are reasonably rare but there are some systematic points that can be made about when internal concurrences occur. The rules for central concurrence are perhaps the easiest to see, as well as when concurrences occur along a diameter of isosceles triangles images. These are examined elsewhere.

Here we focus attention on ***n*** = 12 for a couple of reasons. First, we have often used ***n*** = 12 for introducing ideas because we can build off peoples familiarity with clockfaces. Second, we have exact coordinates for the vertices of a 12-gon using ± versions of four numbers: 0, 0.5, $\sqrt{3}/2$ = 3^0.5/2 = 0.866…, 1, instead of resorting to trigonometric functions.

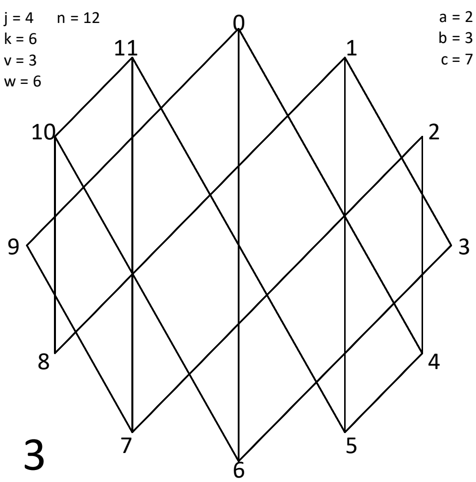


1

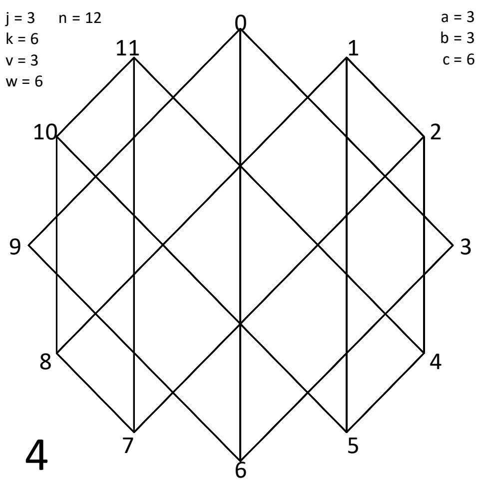
We know that there are two versions of ***a***, ***b***, ***c*** triangles images for each ***a***, ***b***, ***c*** triple when ***n*** = ***a***+***b***+***c*** is even (VT and no VT). Additionally, internal apexes require ***a*** > 1. There are 7 triples with ***a*** > 1 and ***n*** = 12, so 14 images must be visually checked for internal apexes. Eight of these images have internal points of concurrence. Images 1 at 2,2,8 VT and 2 at 2,4,6 VT have a single central point of concurrence. The other six have off-center points of concurrence that come in pairs due to the 180° rotational symmetry inherent in even ***n*** images.



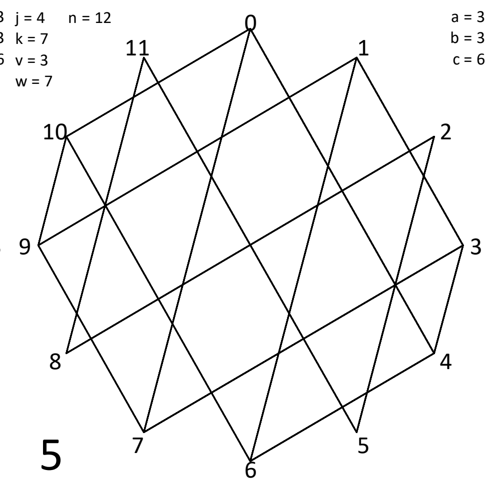
2



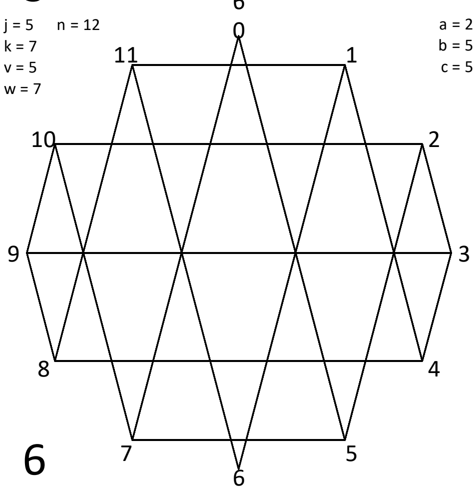
3



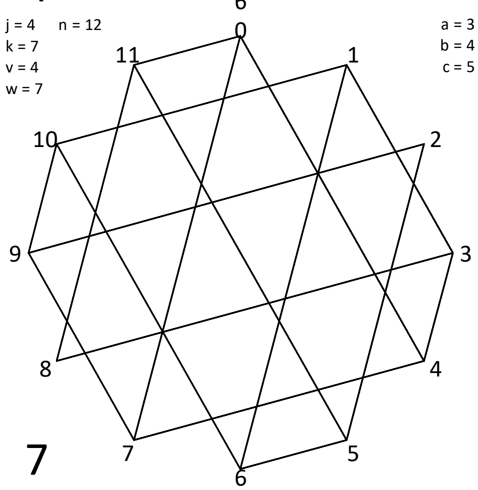
4



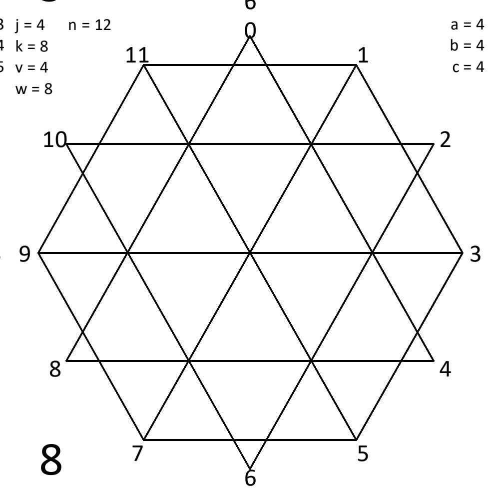
5



6

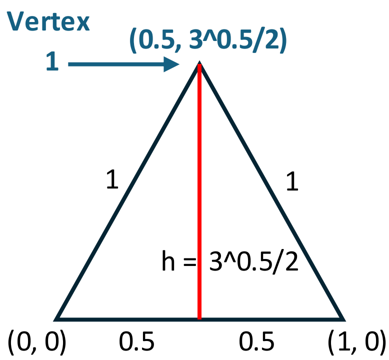


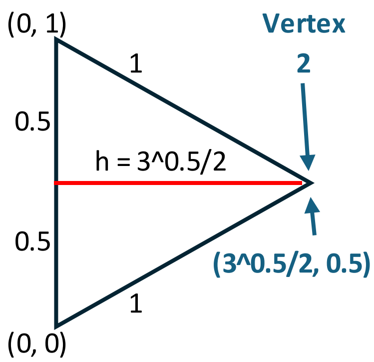
7



8

Four of the eight images (1, 4, 6 and 8) have lines of symmetry that are also lines in the image. Each is based on isosceles triangles, although 8 is equilateral. All the internal apexes on these line of symmetry diameters are points of concurrence. These lines of symmetry coincide with some of the baselines of the isosceles triangles. Note that the equilateral triangles image has three such lines of symmetry (1-7, 3-9, and 5-11) so that there is a total of 7 internal points of concurrence and only the first and last apexes on the first and third arcs are not points of concurrence.





MA Detail. Any apparent concurrence can be checked using the Checking for Concurrence file that is based on trigonometric values for the coordinates of the vertices of the ***n***-gon. But since a 12-gon is based on equilateral triangles of side length 1 and height 3^0.5/2 we can readily obtain the coordinates of the 12-gon's vertices without resorting to trigonometry.

The two equilateral triangles show how the coordinates are obtained simply from what we know about the attributes of those triangles. If one side and hypotenuse of a right triangle are 0.5 and 1 then the other side must be 3^0.5/2 because $x^2+y^2 = 1$ on the unit circle.

Vertices in quadrant IV (4 and 5) have the same x value and minus the y value of those in quadrant I. And vertices quadrants II and III have minus the x value but the same y value as those in quadrants I and IV. The table to the right shows the resulting vertices.

12-gon Vertices

#	x	y	Quad.
0	0	1	I/II
1	0.5	0.866	I
2	0.866	0.5	I
3	1	0	I/IV
4	0.866	-0.5	IV
5	0.5	-0.866	IV
6	0	-1	III/IV
7	-0.5	-0.866	III
8	-0.866	-0.5	III
9	-1	0	II/III
10	-0.866	0.5	II
11	-0.5	0.866	II

Note: 3^0.5/2 = 0.866...

One can use these vertices to create equations for lines connecting any two vertices. These equations can be used to verify that the intersections seen in images 1-8 are indeed points of concurrence.

The table below summarizes the internal concurrence information. Unsurprisingly, each non-central point of concurrence involves some variation on 3^0.5. If you do this exercise on your own, it is worth noting that the simplified equations provided in the table are not the only way you are likely to see a specific value. (For example, in finding the intersection in image 7, the solution x = 1/(1+3^0.5) was found by intersecting lines (0,4) with (1,6) but x = (1+3^0.5)/(4+2*3^0.5) was found by intersecting (0,4) with (2,9). Both simplify to (3^0.5-1)/2 = 3^0.5/2-0.5.)

Location of Concurrent Interior Apexes in Quadrants I or IV given ***n*** = 12							Attributes of Interior Apex		
See	Image type		x coordinate		y coordinate		Distance	Line	Interior angle
image	***a,b,c***	style	value	equation	value	equation	from origin	lengths*	distribution^
1	2,2,8	VT	0		0		0	6 x 6 x 6	1/1, 1/1, 4,4
2	2,4,6	VT	0		0		0	6 x 6 x 6	1/1, 2/2, 3/3
8	4,4,4	VT	0		0		0	6 x 6 x 6	2/2, 2/2, 2/2
3	2,3,7	no VT	0.5		0.1339746	=1-3^0.5/2	0.5176	5 x 4 x 4	1/1, 1/2, 2/5
4	3.3,6	VT	0		0.3660254	=3^0.5/2-0.5	0.3660	5 x 5 x 6	1/2, 1/2, 3/3
5	3,3,6	no VT	0.6830127	=(3^0.5+1)/4	-0.1830127	=(1-3^0.5)/4	0.7071	4 x 4 x 3	1/2, 1/2, 1/5
6	2,5,5	VT	0.26794919	=2-3^0.5	0		0.2679	6 x 5 x 5	1/1, 2/3, 2/3
6	2,5,5	VT	0.73205081	=3^0.5-1	0		0.7321	6 x 3 x 3	1/1, 1/4, 1/4
7	3,4,5	VT	0.3660254	=3^0.5/2-0.5	0.3660254	=3^0.5/2-0.5	0.5176	5 x 4 x 5	1/2, 1/3, 2/3
8	4,4,4	VT	0.28867513	=3^0.5/6	0.5		0.5774	4 x 4 x 6	1/3, 1/3, 2/2
8	4,4,4	VT	0.57735027	=3^0.5/3	0		0.5774	6 x 4 x 4	2/2, 1/3, 1/3

* Line length measured as number of vertices spanned in the direction opposite angles ***a*** , ***b*** , ***c*** .

^ Interior angle distribution is number of vertices spanned in the ***a*** , ***b*** , ***c*** directions at apex point.

One final point is that the distance from the origin to concurrent points are sometimes the same. This is unsurprising for the equilateral triangles image 8. But the concurrences in images 3 and 7 are both 0.5176 from the center. It is easy to see why by staring at each image for a moment. Both concurrent points are on the line (0,4). Consider superimposing a line to 3 from (2,9) or to 7 from (2,7). Both create a 2-7-9 triangle. This triangle is isosceles and the base from 7 to 9 is parallel to (0,4) thus the intersection points on (0,4) are the same distance from the origin in the two images.

This commonality of distance to the center is more readily understood by noting that the concurrences in images 3 and 7 are both based on jumps of 4 and 5 (as noted by the line lengths in the second to right column). This stacked star, created using the *Stacked Stars Excel* file in **PwP** shows the 12 points of 4 line concurrence that occur when one superimposes a 12,4 star on a 12,5-star. Two lines span 4 vertices and two lines span 5 at each 4-lines intersection point.

Each concurrence is midway between vertices of the dodecagon. Call these half-way-between concurrences 0.5, 1.5, … , 11.5 thinking of the hour-hand on a clock. In this visualization, the concurrence in image 3 between lines 0-4, 1-5 and 2-7 is at 2.5 and the one from lines 6-10, 7,11 and 8-1 are at 8.5. Both are the intersection of 2 lines of length 4 and one of length 5. By contrast, image 7 has two lines of length 5 (9-2 and 1-6) and one of length 4 (0-4) concurrent at 1.5 and its rotational counterpart concurrent at 7.5. Whether two lines have length 4 and one has length 5 or the reverse, the result is the same, the concurrence occurs at the same distance from the origin due to the concurrence of these four lines.