Concurrence for Triangles Images for n less than 25

Table 2 presents the results of a comprehensive review of triangles images for n < 25. This review was prompted by an interest in knowing how many images are possible for a given n and how many of those images had internal points of concurrence. Along the way, certain regularities were observed, and it was determined that the first question was a bit deeper than initially believed.

1*	2	3	4	5	6	7	8	9	10	11
Table 1		Total Images,		Non-		Concurrent Images				
		Tot., is # odd n,		Concurrent		Type of				
	#	2# for even n		Images		Concurrence			Images	Fewer Δs
n	a,b,c	Tot.(n)	Tot. ≤ n	a = 1	a > 1	C	oC	D		
3	1	1	1	1	0	-	-	-		
4	1	2	3	2	0	-	-	-		
5	2	2	5	2	0	-	-	-		
6	3	6	11	4	1	1	-	**3**	1	1
7	4	4	15	3	1	-	-	-		
8	5	10	25	6	2	1	1	2	2	3
9	7	7	32	4	3	-	-	-		
10	8	16	48	8	5	2	1	3	3	6
11	10	10	58	5	5	-	-	-		
12	12	24	82	10	6	3	5	**6**	8	21
13	14	14	96	6	8	-	-	-		
14	16	32	128	12	13	4	3	5	7	18
15	19	19	147	7	12	-	-	-		
16	21	42	189	14	18	5	5	6	10	27
17	24	24	213	8	16	-	-	-		
18	27	54	267	16	14	7	28	**9**	24	89
19	30	30	297	9	21	-	-	-		
20	33	66	363	18	32	8	8	8	16	48
21	37	37	400	10	27	-	-	-		
22	40	80	480	20	42	10	8	9	18	58
23	44	44	524	11	33	-	-	-		
24	48	96	620	22	36	12	42	**12**	38	148

* Columns are numbered 1-11 to simplify discussion.

How many images are possible for a given n? Table 1 shows the number of images for a given n. As we have seen elsewhere, any three non-parallel lines connecting vertices of a regular n-gon create triangles images with angles spanning 1 ≤ a ≤ b ≤ c and a+b+c = n vertices.

To systematically count the number of a,b,c alternatives for a given n the choice was made to list alternatives starting with the largest c = n-2 and continuing to the smallest c = INTEGER(n/3)+1 for n not divisible by 3 and c = n/3 if n is divisible by 3. Thereafter, a and b are listed with a increasing from 1 and b decreasing from n-c-1 to the point where a = b or a = b-1 in order to keep a+b = n-c while maintaining a ≤ b ≤ c.

Since these instructions may be difficult to follow at an abstract level, consider the 48 a,b,c triples for n = 24, the right-most set of images listed in Table 2. This set of triples was chosen because it has the longest span of values showing how the pattern of angles is created before that pattern is broken at c = 11, highlighted in green. Prior to that, note that the pattern for c is that c decreases by 1 from 22 = n-2 in increasingly long steps.

Step lengths come in pairs before increasing by 1 in length: 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6 (or 36 a,b,c triples) before c = 11 is reached. The smallest a for n = 24 given c = 11 is a = 2 since otherwise b > c since a+b = 13 = n-c. From there, each additional drop in c requires an increase of the smallest a by 2 (for the same reason). Note that in each such instance, the resulting triangles image is isosceles since b = c. Note also that the green highlighted c values decline by 1 for each drop in n of 2 but the same pattern for a occurs for these values of n.

One benefit of these highlighted values is that the same a,b pairs that only differ by c = n-a-b are in the same row across n values (for example: a,b = 4,4 is seen in the same row for n = 16, 20, 22, and 24 since 4,4,8 is above the isosceles 2,7,7 for n = 16 but 2,7,n-9 is 1 row lower than 2,7,7 because 1,8 is in that row for larger n (and 1,8,7 violates b ≤ c and 1,7,8 is already listed above). Note also that a = 1 values in Table 2 are highlighted grey since these images lack internal apexes.

The number of images follows the pattern shown in columns 2 and 3 of Table 1. This number is based on the number of partitions of n into 3 positive parts. The sequence in column 2 is in *Sloane's On-Line Encyclopedia of Integer Sequences* (A069905). But this is only part of the way to our answer because we know that there are two versions when n is even (VT and no VT) but only one when n is odd. The sequence in column 3 is not recognized by Sloane's but is easily obtained from the sequence noted above.

How common is concurrence? Of the 620 images with n < 25 (last entry in Table 1, column 4), 198 (sum of column 5) have a = 1 and therefore have no internal apexes. 30% of remaining images (127 (the sum of column 10) of 422 = 620-198 with a > 1) have points of concurrence. These points of concurrence are not, however, evenly distributed across n.

Patterns of Concurrence. Several patterns are visible in Table 2 which categorizes various types of concurrence that occur in triangles images for ***n*** < 25.

Odd *n*. The most stunning pattern, of course, is that there are no concurrences when ***n*** is odd for ***n*** < 25. Of the 192 odd ***n*** images, 66 have ***a*** = 1 and the remaining 126 have ***a*** > 1 but none of these images have points of concurrency.

[MA. I asked Nate Stambaugh, a self-described journeyman mathematician, about this pattern and he was able to confirm that this was indeed true for any odd ***n***. He did this by pointing me to a 1998 article by Poonen and Rubinstein, *"The Number of Intersection Points Made by the Diagonals of a Regular Polygon,"* **SIAM Journal of Discrete Mathematics**, V 11, no. 1, 133-156 that answers a different question. That question is sufficiently interesting and relevant to our situation that I provide this quote from the first paragraph of that paper:

> "… It will result from our analysis that for *n* > 4, the maximum number of diagonals of a regular *n*-gon that meet at a point other than the center is
> 2 if *n* is odd,
> 3 if *n* is even but not divisible by 6,
> 5 if *n* is divisible by 6 but not 30, and,
> 7 if *n* is divisible by 30,
> with two exceptions: this number is 2 if *n* = 6 and 4 if *n* = 12."

We will examine these diagonal intersection points in more detail because they help explain the results seen in Table 2. Additionally, these results are inherently interesting and will be explored for a variety of ***n*** including ***n*** = 18, 24, and 30.**]**

The first part of their statement confirms that the results obtained for odd ***n*** from 5 to 23 extends to all odd ***n***. No three lines from vertex to vertex meet at a single point in the interior of a regular ***n***-gon if ***n*** is odd.

Additional patterns of concurrence are readily discernable in Table 2. Because of the above observation, each is, by necessity, an even ***n*** pattern. These patterns are initially mentioned here but are examined in greater detail in subsequent sections.

VT/no VT. This pattern may not hold for larger ***n***, but for ***n*** < 25 we see that the only ***n*** for which both VT and no VT images have concurrences are 12, 18, and 24, each is divisible by 6. In each of those instances, more than half of ***a*** > 1 images (71/128 = 55.5%) have points of concurrence. By contrast, for ***n*** = 2 or 4 mod 6, one third of ***a*** > 1 images (56/168 = 33.3%) have points of concurrence.

The version of even ***n*** images that have concurrences alternates between VT and no VT. When ***n*** = 0 mod 4, points of concurrence are seen in VT images and when ***n*** = 2 mod 4 concurrences are seen in the no VT version. One of the reasons for this flipping back and forth is due to what is required for a central concurrence to occur.

Types of concurrence. Concurrences are categorized into three overlapping types: C, oC, and D (with summary information in columns 7-9 of Table 1).

C is a central concurrence, and it happens when the image has 3 diameter lines (column 7).
oC is an off center concurrence. It occurs when the concurrent point is not at the center of the image (column 8).
D requires that one of the lines in the concurrence be a diameter and all intersections on that line are concurrences (column 9).

C and oC can occur individually, as well as in combination. For example, C and oC concurrences are also often D so C D and oC D entries are common in Table 2. Any D concurrence will by necessity be either C or oC depending on whether the middle-sized angle, ***b***, is even or odd. But C and oC concurrences are possible without D and they are possible to occur together such as the C oC 4,5,8 no VT image with 3 concurrences, 2 that are oC (since oC concurrences come in pairs) and 1 that is C.

Additionally, we see C D images that also include one or more oC concurrences (such as the ***n*** = 24, 6,6,12 VT image which is listed as C D+3oC. It has 11 concurrence points, 5 on the diameter (the C D part) and 6 from the 3 oC concurrences.

2. Concurrence for Triangles Images, 3 ≤ *n* ≤ 24

a	*b*	*c*	VT C/oC/D	VT Tot.	no VT C/oC/D	no VT Tot.
6	*n,#*	3				
1	1	4				
1	2	3				
2	2	2			C 3D	1
8	*n,#*	5				
1	1	6				
1	2	5				
1	3	4				
2	2	4	C D	1		
2	3	3	oC D	2		
10	*n,#*	8				
1	1	8				
1	2	7				
1	3	6				
2	2	6			C D	1
1	4	5				
2	3	5	**Smallest no concurrence.**			
2	4	4			C D	3
3	3	4			oC D	2
12	*n,#*	12	See the previous section, Images 1-8.			
1	1	10				
1	2	9				
1	3	8				
2	2	8	**1.** C D	1		
1	4	7				
2	3	7			**3.** oC	2
1	5	6				
2	4	6	**2.** C	1		
3	3	6	**4.** oC D	2	**5.** oC	2
2	5	5	**6.** oC D	4		
3	4	5	**7.** oC	2		
4	4	4	**8.** C 3D	7		
14	*n,#*	16	See next section, Figure 4.			
1	1	12				
1	2	11				
1	3	10				
2	2	10			C D	1
1	4	9				
2	3	9				
1	5	8				
2	4	8			C	1
3	3	8			oC D	2
1	6	7				
2	5	7				
3	4	7				
2	6	6			C D	5
3	5	6			oC	2
4	4	6			C D	3
4	5	5			oC D	4

a = 1. ***a*** > 1 and no concurrence.

a	*b*	*c*		VT C/oC/D	VT Tot.		no VT C/oC/D	no VT Tot.
16	*n,#*	21	See next section, Figure 4.					
1	1	14						
1	2	13						
1	3	12						
2	2	12		C D	1			
1	4	11						
2	3	11						
1	5	10						
2	4	10		C	1			
3	3	10		oC D	2			
1	6	9						
2	5	9						
3	4	9						
1	7	8						
2	6	8		C	1			
3	5	8						
4	4	8		C D	3			
2	7	7		oC D	6			
3	6	7		oC	2			
4	5	7		oC	2			
4	6	6		C D	5			
5	5	6		oC D	4			
18	*n,#*	27	See next section, Figures 1-4.					
1	1	16	Figure 1			Figure 1		
1	2	15						
1	3	14						
2	2	14					C D	1
1	4	13						
2	3	13		oC	2			
1	5	12						
2	4	12		oC	2		C	1
3	3	12					oC D	2
1	6	11						
2	5	11		oC	2			
3	4	11	1.2	2oC	4		oC	2
1	7	10						
2	6	10		oC	2		C	1
3	5	10		2oC	4		oC	2
4	4	10				1.8	C D+2oC	7
1	8	9						
2	7	9						
3	6	9						
4	5	9						
2	8	8				1.1	C D	7
3	7	8		2oC	4	1.6	oC	2
4	6	8	1.5	2oC	4		C + oC	3
5	5	8	1.4	4oC	8	1.9	oC D	4
4	7	7		2oC	4	1.3	oC D	6
5	6	7					oC	2
6	6	6				1.7	C 3D	13

a	*b*	*c*	VT C/oC/D	VT Tot.
20	*n,#*	33		
1	1	18		
1	2	17		
1	3	16		
2	2	16	C D	1
1	4	15		
2	3	15		
1	5	14		
2	4	14	C	1
3	3	14	oC D	2
1	6	13		
2	5	13		
3	4	13		
1	7	12		
2	6	12	C	1
3	5	12		
4	4	12	C D	3
1	8	11		
2	7	11		
3	6	11		
4	5	11		
1	9	10		
2	8	10	C	1
3	7	10		
4	6	10	C	1
5	5	10	oC D	4
2	9	9	oC D	8
3	8	9	oC	2
4	7	9	oC	2
5	6	9	oC	2
4	8	8	C D	7
5	7	8	oC	2
6	6	8	C D	5
6	7	7	oC D	6

a	*b*	*c*	no VT C/oC/D	no VT Tot.
22	*n,#*	40		
1	1	20		
1	2	19		
1	3	18		
2	2	18	C D	1
1	4	17		
2	3	17		
1	5	16		
2	4	16	C	1
3	3	16	oC D	2
1	6	15		
2	5	15		
3	4	15		
1	7	14		
2	6	14	C	1
3	5	14		
4	4	14	C D	3
1	8	13		
2	7	13		
3	6	13		
4	5	13		
1	9	12		
2	8	12	C	1
3	7	12		
4	6	12	C	1
5	5	12	oC D	4
1	10	11		
2	9	11		
3	8	11		
4	7	11		
5	6	11		
2	10	10	C D	9
3	9	10	oC	2
4	8	10	C	1
5	7	10	oC	2
6	6	10	C D	5
4	9	9	oC D	8
5	8	9	oC	2
6	7	9	oC	2
6	8	8	C D	7
7	7	8	oC D	6

a	*b*	*c*	VT C/oC/D	VT Tot.	no VT C/oC/D	no VT Tot.
24	*n,#*	48				
1	1	22				
1	2	21				
1	3	20				
2	2	20	C D	1		
1	4	19				
2	3	19				
1	5	18				
2	4	18	C	1		
3	3	18	oC D	2		
1	6	17				
2	5	17				
3	4	17	oC	2		
1	7	16				
2	6	16	C	1		
3	5	16				
4	4	16	C D	3		
1	8	15				
2	7	15			oC	2
3	6	15	oC	2	oC	2
4	5	15	oC	2	2oC	4
1	9	14				
2	8	14	C	1		
3	7	14	oC	2	oC	2
4	6	14	C + 2oC	5		
5	5	14	oC D	4	2oC	4
1	10	13				
2	9	13			oC	2
3	8	13				
4	7	13			2oC	4
5	6	13			oC	2
1	11	12				
2	10	12	C	1		
3	9	12	oC	2		
4	8	12	C	1		
5	7	12				
6	6	12	C D+3oC	11		
2	11	11	oC D	10		
3	10	11	oC	2		
4	9	11	3oC	6		
5	8	11	oC	2		
6	7	11	2oC	4		
4	10	10	C D+2oC	13		
5	9	10	2oC	4	oC	2
6	8	10	C + oC	3		
7	7	10	oC D	6	2oC	4
6	9	9	oC D	8		
7	8	9	oC	2		
8	8	8	C 3D	19		

First ***c*** for ***a*** > 1 for each ***n***.

Fill Color & Acronym Guide

a,b,c are Δ angles with ***a*** ≤ ***b*** ≤ ***c***, ***a*** + ***b*** + ***c*** = ***n***.

is number of ***a,b,c*** triples for a given ***n***.

VT is an image with a vertex triangle.

no VT is an image with no vertex triangles.

C is a central concurrence.

oC is a concurrence that is off-center.

D signifies that all intersections on a diameter line are points of concurrence.

Tot. is the number of concurrences in the image.