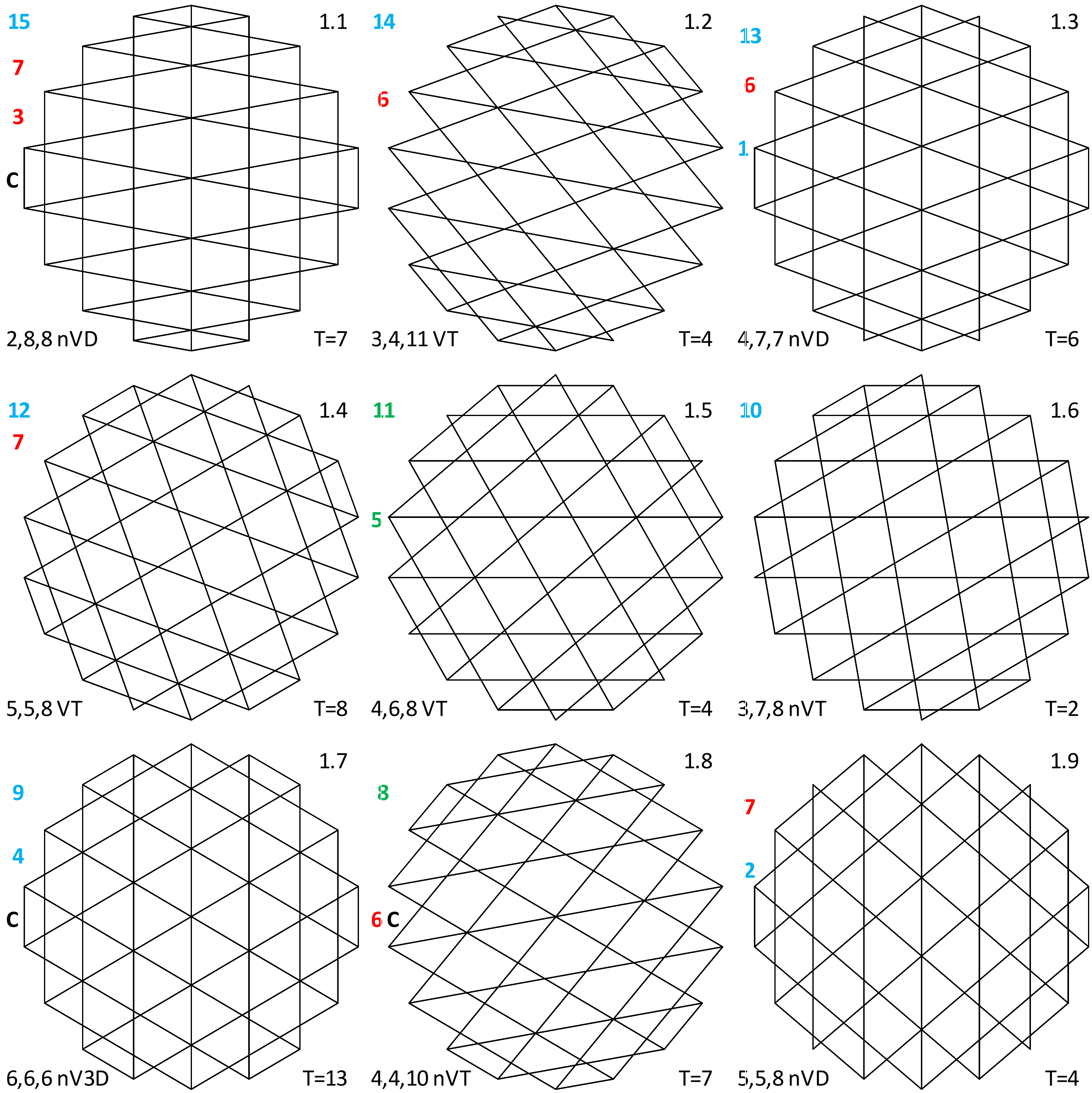
# On the Distance from the Center of Off-Center Concurrences

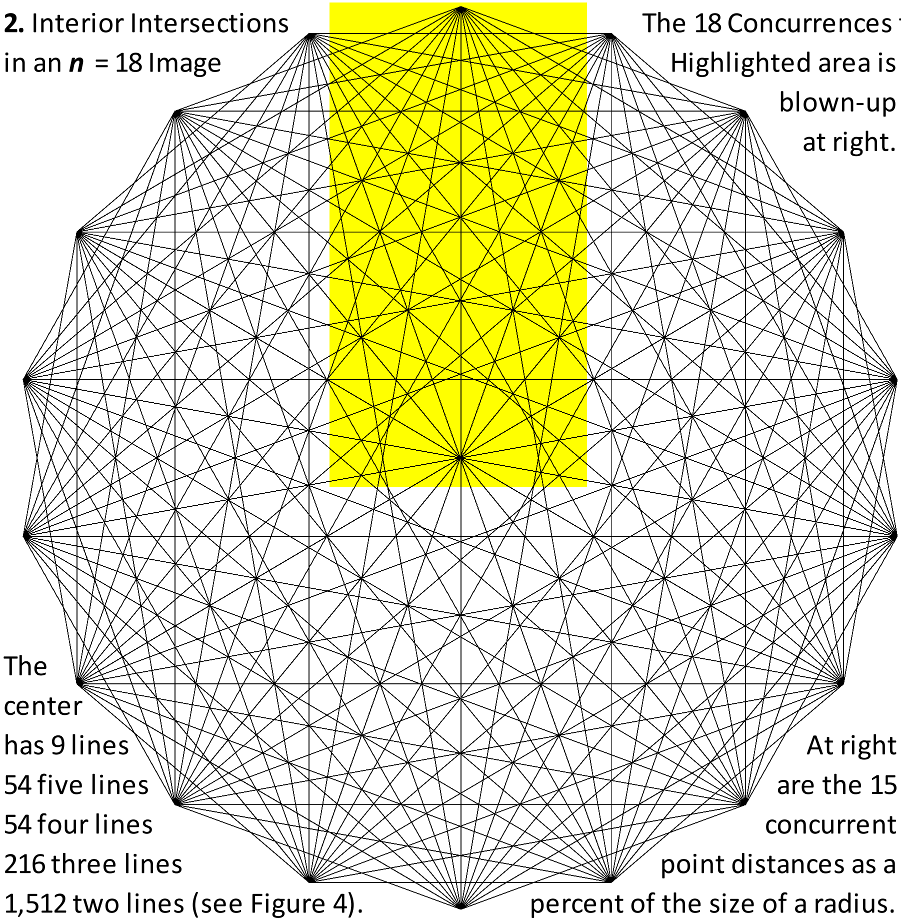
There are 89 concurrences in Table 2 of the last section for ***n*** = 18 but 7 are central concurrences. The other 82 off-center concurrences come in pairs. One might expect that each pair of off-center concurrences is at a different distance from the center. As it turns out, there are only 15 distinct distances, and they are shown in the nine images in Figure 1.

**1.** Concurrent Points at the 15 Distances from the Center for ***n*** = 18: 1 = 18.5% to 15 = 87.9% of the Radius*




* Images have been rotated or reflected so the largest concurrency in the image occurs within 0° to 10° of vertical to match circled blow-up points 7 to 15 from Figure 2. These points are between vertices 0 and 1/2 (halfway between 0 and 1) of the 18-gon. The 9 images are ordered from largest to smallest (from 15 to 7) according to greatest distance concurrency point. Smaller concurrency distances (from 7 to 1) are noted to the left of each image but may not be in the blow-up location. Due to symmetry, each point of concurrency (except center noted as **C**) has a mirror image point of concurrency through the center. Isosceles triangles images with off-diameter concurrences have four points at each distance like 12 and 7 for 1.4 and 8 for 1.8 rather than two.

**2.** Interior Intersections in an ***n*** = 18 Image

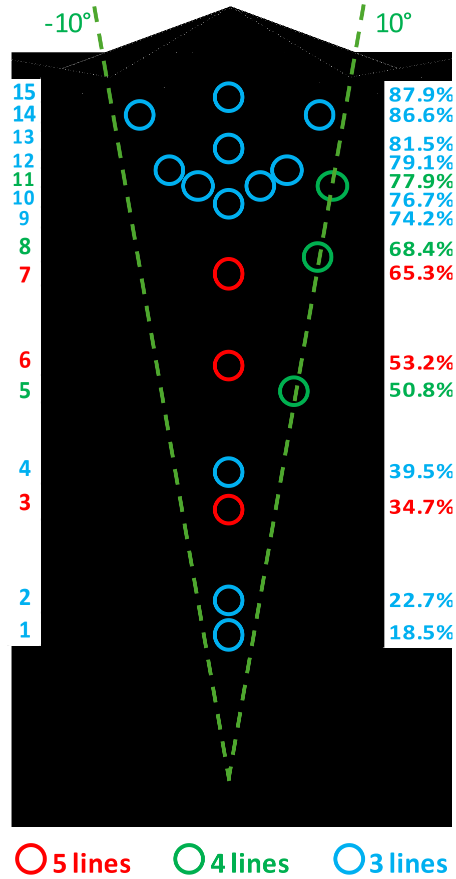


The center has 9 lines
54 five lines
54 four lines
216 three lines
1,512 two lines (see Figure 4).

The 18 Concurrences from -10° to 10° of the Vertical Radius

Highlighted area is blown-up at right.

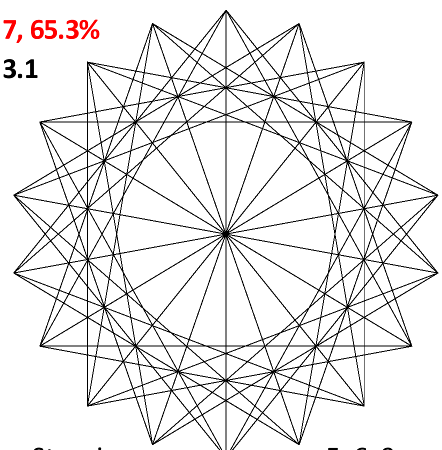
At right are the 15 concurrent point distances as a percent of the size of a radius.




**3.** Stacked Stars showing the 5 lines (top row) and 4 lines (bottom row) Concurrences given ***n*** = 18

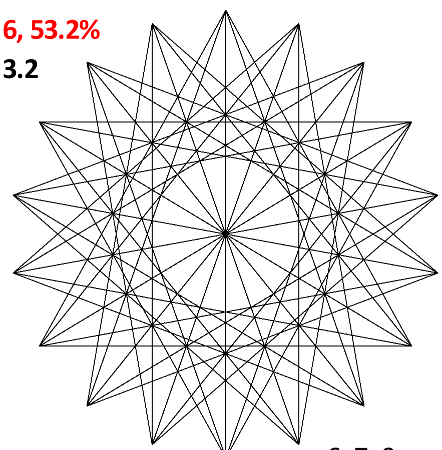
Distance from center

7, 65.3%
**3.1**



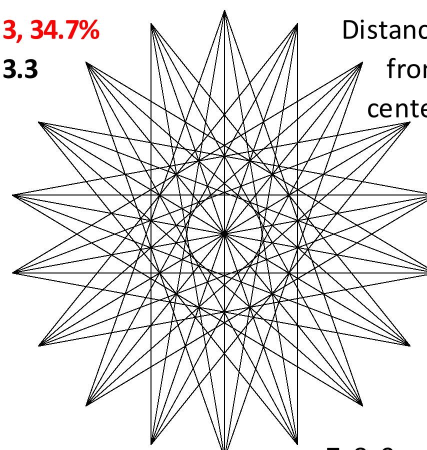
Star sizes 5, 6, 9

6, 53.2%
**3.2**



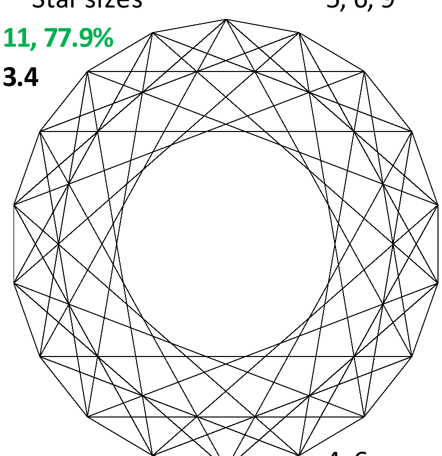
6, 7, 9

3, 34.7%
**3.3**



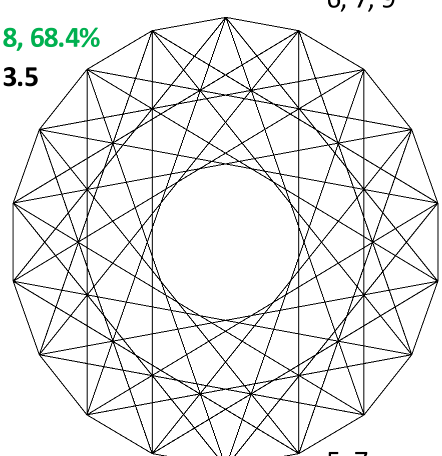
7, 8, 9

11, 77.9%
**3.4**



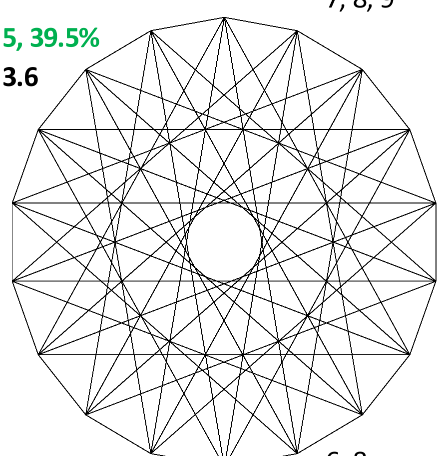
4, 6

8, 68.4%
**3.5**



5, 7

5, 39.5%
**3.6**

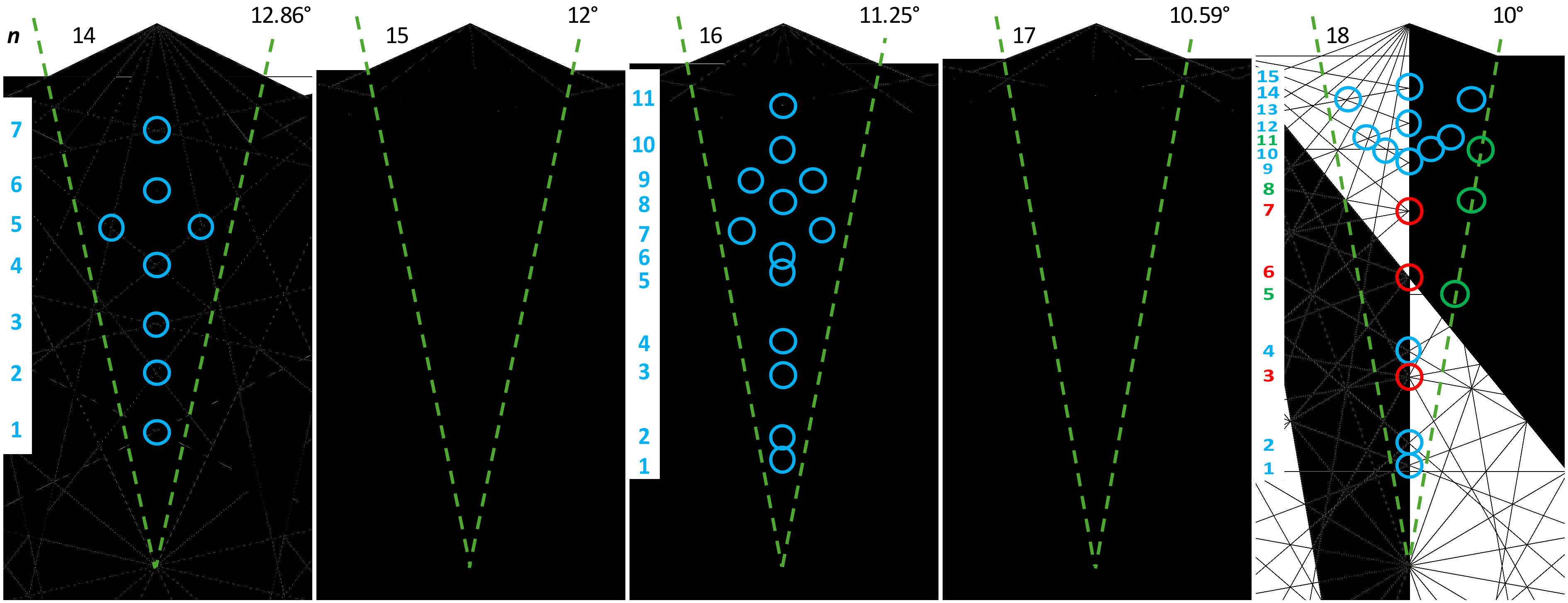


6, 8

**How many lines?** The 15 ***n*** = 18 distances in Figures 1 and 2 are distinct from one another in the number of lines in the concurrence. There are nine distances with **3 lines** concurrences (3 of the 12 are not on either of the two lines of symmetry at 0° and 10°), three with **4 lines** midway between vertices, and three with **5 lines** beneath each vertex. Figure 3 shows the stacked stars associated with these **5 lines** and **4 lines** concurrences (the **4 lines** images also include the 18-gon).

Figure 4 details parts of five mystic roses (less the single jump polygonal lines) which show concurrence points in the 1/***n*** wedge bounded by dashed green lines centered on vertex 0 for ***n*** = 14-18. These values were chosen to highlight the pattern of concurrence that repeats for any ***n*** > 6. When ***n*** is odd, there are no concurrences; when ***n*** = 2 or 4 mod 6, there are **3 lines** concurrences; and when ***n*** = 0 mod 6 there are concurrences of more than 3 lines.

**4.** Counting Diagonal Intersection Points in Regular Polygons by Focusing on Halfway Between Vertices ***n*** -1 and 1, from (-180/***n*** °, 180/***n*** °]*




| | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|
| In wedge: | 46 2 lines<br>8 ○ **3 lines** | 91 2 lines | 73 2 lines<br>13 ○ **3 lines** | 140 2 lines | 84 2 lines<br>12 ○ **3 lines**<br>3 ○ **4 lines**<br>3 ○ **5 lines** |
| Center, ***n***/2: | 1 7 lines | | 1 8 lines | | 1 9 lines |
| Total: | 757 = 14(46+8)+1 | 1,365 = 15·91 | 1,377 = 16(73+13)+1 | 2,380 = 17·140 | 1,837 = 18(84+12+3+3)+1 |

*Due to rotational symmetry, the number of intersections in the final image is ***n*** times those found in a single half-open, half-closed 360/***n***° wedge (the dashed lines) associated with vertex 0 plus 1 for the center when ***n*** is even To avoid double counting, count intersections on one side of the dashed line halfway between vertices. This is why the three four line concurrent points on the 10° dashed line in the far right ***n*** = 18 image are circled in **green** but the symmetric points on the -10° line are not circled. Rank order noted on left for each wedge.

This is the pattern of concurrences noted by Poonen and Rubenstein, quoted in the previous section. They show that regardless of ***n***, there are no concurrences if ***n*** is odd. For even ***n***, the maximum number of lines in a non-central concurrence is 3 if ***n*** is not divisible by 6, 5 if ***n*** is divisible by 6 but not 30, and 7 if ***n*** is divisible by 30. They also noted an exception for ***n*** = 12. In this instance, the maximum number is 4 as we saw in Section 7.2. In that instance, two distinct triangles images 2,3,7 no VT and 3,4,5 VT have a concurrence at 0.5176 from the center. Both concurrences are based on lines spanning 4 and 5 vertices. The first has one 5 and two 4s (and hence is no VT) the second has two 5s and one four (and hence is VT). These are the two possible combinations of three lines, two of which are 4s and two of which are 5s.

The patterns in Figures 2-4 are not, of course, triangles patterns. Nonetheless, they inform our understanding of triangles concurrences by shedding light on why some concurrences occur at the same distance from the center. Before returning to triangles, it is worthwhile to push our understanding of concurrences in this more general setting a bit further.

**Line lengths and lines of symmetry.** There are two lines of symmetry in each wedge: on the vertex radius and on the diameter halfway between vertices.

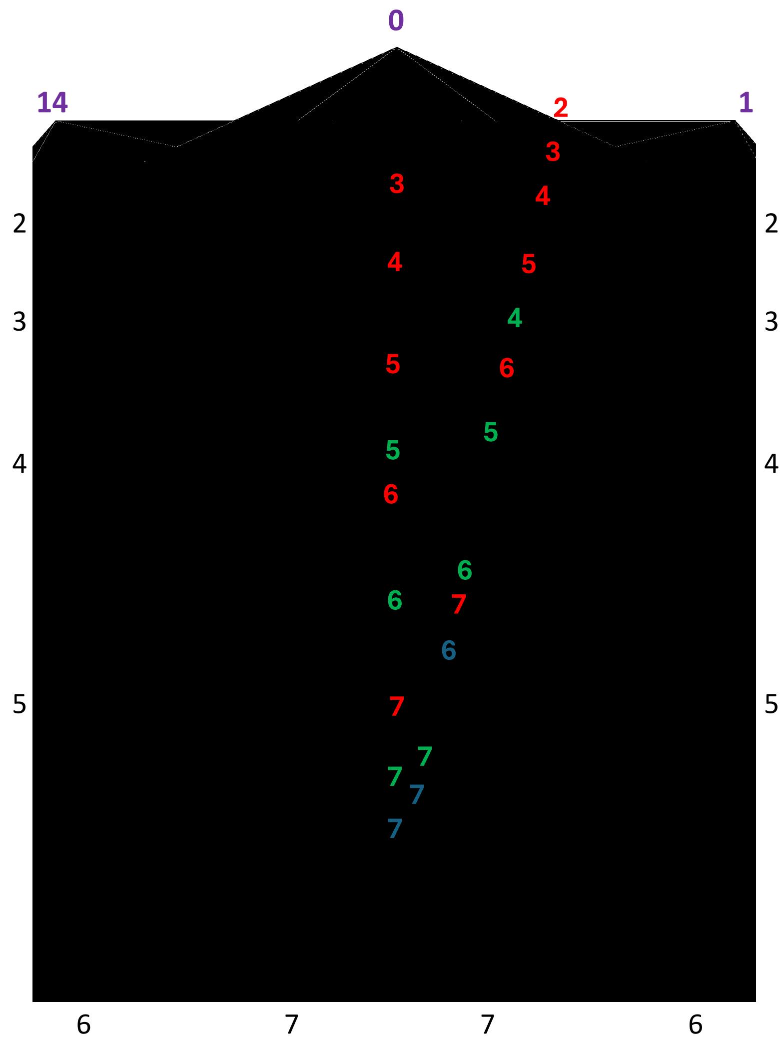
**CLAIM:** *If two lines in a concurrence span the same number of vertices, **J**, then the intersection point must be on one of the two lines of symmetry.*

The reasoning is simple: Due to rotational symmetry, we can assume that one line spans vertices 0***J***. To be an interior intersection, one of the endpoints of the second line must be between vertices 1 and ***J***-1. Let that vertex be ***K*** where 0 < ***K*** < ***J*** and the end of that line will be at vertex ***K***+***J*** or ***K***-***J*** mod ***n***. These two lines will be symmetric about the diameter midway between these two vertices, at (***K***+***J***)/2 or ***K***/2, and the intersection point will be on this diameter. Since ***K*** and ***J*** are both whole numbers, each average is a whole number or half-way between whole numbers.

This point is driven home by examining the paired line intersections of lengths 2 to 7 in Figure 5, the vertex **0** wedge for ***n*** = 15. This wedge has been expanded a bit from its Figure 4 counterpart so that vertices **1** and **14** are also visible to make it easier to see which lines come from which vertices. There are no concurrences (although the triangle near **6** is small with area of 1.1 E-05 using the concurrence checker) and no diameter lines in this image (since ***n*** is odd) so that the paired lines are more readily visible. See bottom notes for additional details on how the paired lines intersections were formed.

*Vertex radius concurrences.* Two lines of the **3 lines** concurrences on the vertex radius in Figure 4 have the same vertex jump size since they are reflected across the vertex line and the third line is the vertical diameter line. Additionally, only an odd number of lines are concurrent along a vertex radius because the lines come in pairs, except for the vertical diameter line unless there is also a horizontal line at the concurrence point spanning an even number of vertices.

**5. *n*** = 15 Paired Line Intersections in the Vertex **0** Wedge




Line lengths from vertex **0** are noted around the perimeter.

**5 intersections on vertex ray 0 use vertex 1 and 14 lines.**

**6 intersections on 0.5 diameter use vertex 0 and 1 lines.**

**3 intersections on vertex ray 0 use vertex 2 and 13 lines.**

**4 intersections on 0.5 diameter use vertex 14 and 2 lines.**

**Intersection 7 on vertex ray 0 uses lines 3-11 and 12-4.**

**2 intersections on 0.5 diameter use vertex 13 and 3 lines.**

*Half-vertex radius concurrences.* Half-vertex concurrences only occur in pairs due to symmetry about the half-vertex line. This implies that no 3 lines concurrences occur here, and all concurrences must involve an even number of lines. There are three such **4 lines** concurrences in the right panel of Figure 4 on the ***n*** = 18 ray at 10°, midway between vertices 0 and 1. (Theoretically, there could be an odd number of lines in a half-vertex concurrence if that line was also a perpendicular bisector of the half-vertex radius. Such a line would have to span an odd number of vertices.)

*Concurrences not on the vertex radius or half-vertex radius.* Concurrences that are not on the vertex or half-vertex radius come in pairs, one on either side of the vertex radius. The number of vertices spanned by the lines in each such concurrence must be distinct from one another and range from 2 to ***n***/2-1. There is one pair of these points for ***n*** = 14 (spanning 4, 5, 6 vertices), 2 for ***n*** = 16 (spanning 4, 5, 7, and 5, 6, 7 vertices) and 3 for ***n*** = 18 (spanning 3, 4, 5; 4, 5, 7; and 4, 5, 8 vertices). One might conjecture there are (***n***-12)/2 pairs of such points more generally for even ***n*** larger than 12. A quick check of ***n*** = 20 finds that there are indeed 4 such distinct 3 lines concurrent points with each spanning different numbers of vertices. *It is worth noting that each pair is associated with one triangles image concurrence since the other part of the pair is simply the reflected version of that same image.*

**Distance from the Center.** Table 1 provides information about distance from the center for concurrent images originally examined in Table 2 from the previous section for ***n*** = 12 and Figure 4 in this section. The oC triangles image concurrences outlined there match up with the interior intersections of 3 or more lines for ***n*** = 12, 14, 16, and 18.

**Table 1.** Off-Center Triangles Concurrences by Distance from the Center, ***n*** = 12, 14, 16, and 18

| | Type of Concurrence* | | | | | Distance from the Center | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ***a*** | ***b*** | ***c*** | **C/oC/D** | **Tot.** | **Fraction of Radius** | | | **Rank**** | | |
| | **12** | ***n*,#** | **12** | **no VT** | | ***n*** = 12 images **1-8** in Section 7.2. | | | | | |
| | 2 | 3 | 7 | **3.** oC | 2 | 0.518 | | | 3 | | |
| | 3 | 3 | 6 | **5.** oC | 2 | 0.707 | | | 5 | | |
| | **14** | ***n*,#** | **16** | **no VT** | | See Figure 4, ***n*** = 14 wedge. | | | | | |
| | 2 | 2 | 10 | C D | 1 | | | | | | |
| | 2 | 4 | 8 | C | 1 | | | | | | |
| | 3 | 3 | 8 | oC D | 2 | 0.357 | | | 2 | | |
| | 2 | 6 | 6 | C D | 5 | 0.445 | 0.802 | | 3 | 7 | |
| | 3 | 5 | 6 | oC | 2 | 0.629 | | | 5 | | |
| | 4 | 4 | 6 | C D | 3 | 0.555 | | | 4 | | |
| | 4 | 5 | 5 | oC D | 4 | 0.247 | 0.692 | | 1 | 6 | |
| Figure 1 | **18** | ***n*,#** | **27** | **no VT** | | See Figure 4, ***n*** = 18 wedge. | | | | | |
| | 2 | 2 | 14 | C D | 1 | | | | | | |
| | 2 | 4 | 12 | C | 1 | | | | | | |
| | 3 | 3 | 12 | oC D | 2 | 0.347 | | | 3 | | |
| | 3 | 4 | 11 | oC | 2 | 0.532 | | | 6 | | |
| | 2 | 6 | 10 | C | 1 | | | | | | |
| | 3 | 5 | 10 | oC | 2 | 0.653 | | | 7 | | |
| 1.8 | 4 | 4 | 10 | C D+2oC | 7 | 0.532 | 0.684 | | 6 | 8 | |
| 1.1 | 2 | 8 | 8 | C D | 7 | 0.347 | 0.653 | 0.879 | 3 | 7 | 15 |
| 1.6 | 3 | 7 | 8 | oC | 2 | 0.767 | | | 10 | | |
| | 4 | 6 | 8 | C + oC | 3 | 0.684 | | | 8 | | |
| 1.9 | 5 | 5 | 8 | oC D | 4 | 0.227 | 0.653 | | 2 | 7 | |
| 1.3 | 4 | 7 | 7 | oC D | 6 | 0.185 | 0.532 | 0.815 | 1 | 6 | 13 |
| | 5 | 6 | 7 | oC | 2 | 0.347 | | | 3 | | |
| 1.7 | 6 | 6 | 6 | C 3D | 13 | 0.395 | 0.742 | | 4 | 9 | |
| | ***a,b,c*** images with both **VT** and **no VT** concurrences. | | | | | | | | | | |

| | Type of Concurrence* | | | | | Distance from the Center | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | ***a*** | ***b*** | ***c*** | **C/oC/D** | **Tot.** | **Fraction of Radius** | | | **Rank**** | | |
| | **12** | ***n*,#** | **12** | **VT** | | | | | | | |
| | 2 | 2 | 8 | **1.** C D | 1 | | | | | | |
| | 2 | 4 | 6 | **2.** C | 1 | | | | | | |
| | 3 | 3 | 6 | **4.** oC D | 2 | 0.366 | | | 2 | | |
| | 2 | 5 | 5 | **6.** oC D | 4 | 0.268 | 0.732 | | 1 | 6 | |
| | 3 | 4 | 5 | **7.** oC | 2 | 0.518 | | | 3 | | |
| | 4 | 4 | 4 | **8.** C 3D | 7 | 0.577 | | | 4 | | |
| | **16** | ***n*,#** | **21** | **VT** | | See Figure 4, ***n*** = 16 wedge. | | | | | |
| | 2 | 2 | 12 | C D | 1 | | | | | | |
| | 2 | 4 | 10 | C | 1 | | | | | | |
| | 3 | 3 | 10 | oC D | 2 | 0.351 | | | 3 | | |
| | 2 | 6 | 8 | C | 1 | | | | | | |
| | 4 | 4 | 8 | C D | 3 | 0.541 | | | 5 | | |
| | 2 | 7 | 7 | oC D | 6 | 0.199 | 0.566 | 0.848 | 1 | 6 | 11 |
| | 3 | 6 | 7 | oC | 2 | 0.710 | | | 9 | | |
| | 4 | 5 | 7 | oC | 2 | 0.622 | | | 7 | | |
| | 4 | 6 | 6 | C D | 5 | 0.414 | 0.765 | | 4 | 10 | |
| | 5 | 5 | 6 | oC D | 4 | 0.235 | 0.668 | | 2 | 8 | |
| Figure 1 | **18** | ***n*,#** | **27** | **VT** | | **1 for 3**, **2 for 4**, **6 for 5** lines | | | | | |
| | 2 | 3 | 13 | oC | 2 | 0.653 | | | 7 | | |
| | 2 | 4 | 12 | oC | 2 | 0.779 | | | 11 | | |
| | 2 | 5 | 11 | oC | 2 | 0.347 | | | 3 | | |
| 1.2 | 3 | 4 | 11 | 2oC | 4 | 0.532 | 0.866 | | 6 | 14 | |
| | 2 | 6 | 10 | oC | 2 | 0.508 | | | 5 | | |
| | 3 | 5 | 10 | 2oC | 4 | 0.347 | 0.653 | | 3 | 7 | |
| | 3 | 7 | 8 | 2oC | 4 | 0.347 | 0.532 | | 3 | 6 | |
| 1.5 | 4 | 6 | 8 | 2oC | 4 | 0.508 | 0.779 | | 5 | 11 | |
| 1.4 | 5 | 5 | 8 | 4oC | 8 | 0.653 | 0.791 | | 7 | 12 | |
| | 4 | 7 | 7 | 2oC | 4 | 0.532 | | | 6 | | |

*Type of Concurrence data taken from Table 2 in the previous section.

**Rank order noted at left edge of Figure 4 wedges for ***n*** = 14, 16, 18 from smallest to largest distance from center.

**An Aside on Distinct Distances.** It is worth noting that the concurrence distances from the center do not repeat across the values of ***n*** in Table 1. If you line up the distances for each ***n*** (6, 7, 11, and 15 for ***n*** = 12, 14, 16 and 18) and sort them you will not find any overlaps so that there are 6+7+11+15 = 39 distinct distances. Of course this will not always be the case. After all, at least 6 of the ***n*** = 24 distances will overlap with the ***n*** = 12 distances because the same concurrence values can be obtained by doubling each of the ***JKVW*** values and ***n***. This will produce concurrences at the original concurrence points as well as perhaps new points of as well. (Two examples will suffice in making this point. Doubling the values that produced the 3,4,5 VT Image 7 in Section 7.2 produces a 6,8,10 VT image with the same two 0.518 concurrences and one more point of concurrence (the center). But doing the same thing to the equilateral triangles Image 8 produces 12 more points of concurrence since there are now 3 nested interior regular hexagons in the image, at 0.299, 0.577, and 0.816, the middle of which is the one noted in Table 1 above and seen in Image 8 of Section 7.2.)

**Why 15 distances for *n* =18?** We return to the original question: Why are there only 15 distances from the center for concurrences in ***n*** = 18 triangles images?

As noted in Table 1 from the previous section, there are concurrences in 24 of 54 ***n*** = 18 images with 89 concurrences total. Seven are central concurrences. That leaves 82 that are off-center concurrences. These come in pairs due to the symmetry inherent in even ***n*** images so we might imagine 41 different distances. These 41 pairs use only 15 distances total. Table 2 provides summary information about those images to understand why these concurrences can be seen at a much smaller number of distances than one might expect.

The type of triangles in each image holds one key, the other is held by the fact that, unlike when an even ***n*** is not divisible by 6 (like 14 and 16 in Figure 4) one has 4 and 5 lines concurrences. This information is summarized in Table 2.

Scalene triangles (like 1.2, 1.5 and 1.6) and isosceles triangles diameter concurrences (like 1.1, 1.3, and 1.9) come in pairs. Isosceles triangles concurrences not on diameters or lines of symmetry (like 1.4) come in fours. Equilateral images (like 1.7) have three diameters and hence come in sixes.

Each **3 lines** concurrence distance supports a single image at that distance. By contrast, each **4 lines** concurrence has two distinct images at that distance, and each **5 lines** concurrence has six distinct images at that distance.

The final column in Table 2 summarizes how these 82 off-center concurrences are distributed across the 15 distances. Nine of the distances are seen in a single image because they are **3 lines** concurrences. Six of the nine are paired concurrences but **12** has four concurrences since Figure 1.4 is isosceles, and two distances each (**9** and **4**) have 6 since Figure 1.7 is equilateral. Two images each are based on the three **4 lines** concurrences (**11**, **8**, and **5**). Finally, each of the three **5 lines** concurrences supports 6 distinct images (**7**, **6**, and **3**).

The next two sections provide second looks at **4 lines** and **5 lines** concurrences to understand why one obtains multiple distinct triangles images at these distances.

**Table 2.** The Distribution of 82 ***n*** = 18 off-Center Concurrences

| Distance Information from Table 1 | | Type of Triangles Image | | | | Images total* | Off-Center Concurrence by Distance | |
|---|---|---|---|---|---|---|---|---|
| | | | Isosceles on Diameter | | Equilateral Three | | | |
| Rank | Size | Scalene | Yes | No | Diameters | | Lines | Total^ |
| 15 | 0.879 | | 1 | | | 1 | 3 | 2 |
| 14 | 0.866 | 1 | | | | 1 | 3 | 2 |
| 13 | 0.815 | | 1 | | | 1 | 3 | 2 |
| 12 | 0.791 | | | 1 | | 1 | 3 | 4 |
| 11 | 0.779 | 2 | | | | 2 | 4 | 4 |
| 10 | 0.767 | 1 | | | | 1 | 3 | 2 |
| 9 | 0.742 | | | | 1 | 1 | 3 | 6 |
| 8 | 0.684 | 1 | | 1 | | 2 | 4 | 6 |
| 7 | 0.653 | 3 | 2 | 1 | | 6 | 5 | 14 |
| 6 | 0.532 | 3 | 2 | 1 | | 6 | 5 | 14 |
| 5 | 0.508 | 2 | | | | 2 | 4 | 4 |
| 4 | 0.395 | | | | 1 | 1 | 3 | 6 |
| 3 | 0.347 | 4 | 2 | | | 6 | 5 | 12 |
| 2 | 0.227 | | 1 | | | 1 | 3 | 2 |
| 1 | 0.185 | | 1 | | | 1 | 3 | 2 |
| | | | | | | | Total | 82 |

*Images total is count of images by distance.

^Total is the count of off-center concurrences based on the number of each type of concurrence. Scalene and isosceles diameter concurrences are times 2, isosceles concurrences not on the diameter are times 4, and equilateral concurrences are times 6 (since each is on three diameters).