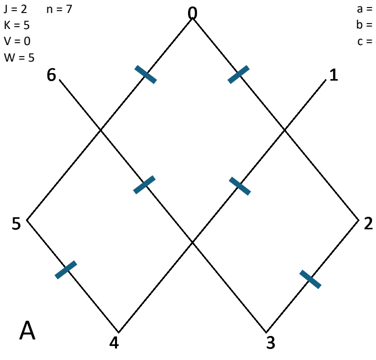
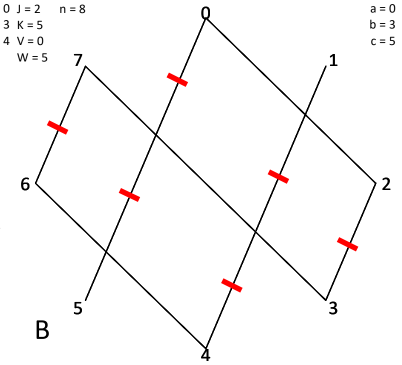
Conditions Ensuring Rhombi in Parallelograms Images

A parallelogram with equal adjacent sides is called a rhombus just like a rectangle with equal adjacent sides is a square. The only time squares occur in rectangles images is when ***n* = 4*k*** but for rhombi in parallelograms images, the rules are more flexible. Some images have rhombi (like A) and some (like B) have none. Here we examine when this occurs.

A versus B. The line segments on opposing sides of a parallelogram are the same size. Therefore, the segments marked with **blue hash marks** in A are all the same size because each is the distance between vertices which are the same because they are vertices of a regular 7-gon (so length 2-3 is the same as 4-5). As a result, the parallelogram with apex at 0 is a rhombus in A.



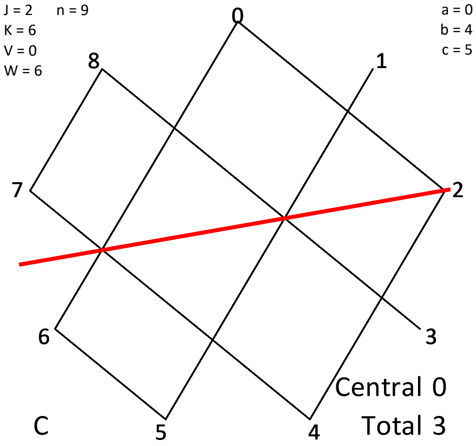


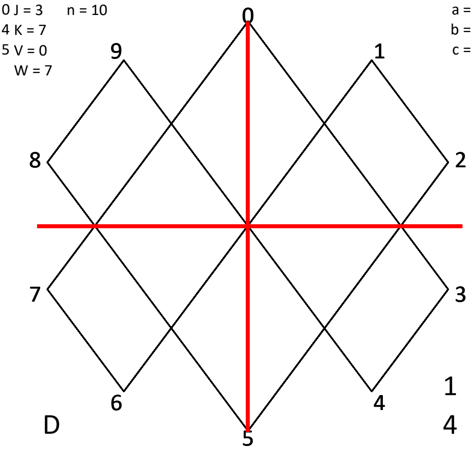
By contrast, the parallelogram with apex at 0 in B is not a rhombus even though it looks like it might be. Here the downward sloping line segment is larger than the upward sloping segments (marked with **red hash marks**) and it is easy to see why: the angle created by connecting vertices 0-1-4 is a right angle and the hypotenuse of a right triangle is larger than either of the legs. The leg 0-1 is the same size as 2-3 so we know adjacent sides are not the same size. (A final thing to note is that the **red** one-vertex-span segments (between vertices of the 8-gon in B) are smaller than the **blue** one-vertex-span segments of the 7-gon in A.)

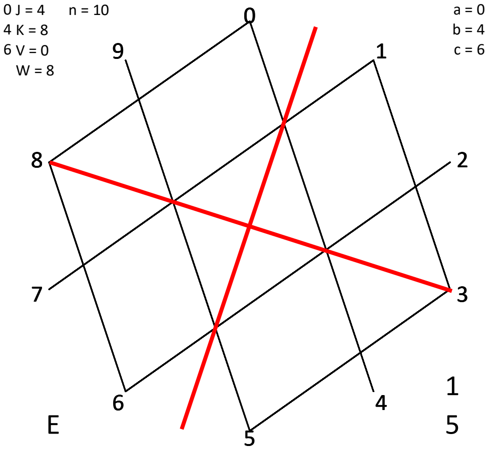
Lines of Symmetry in Parallelograms Images. The lines creating a rhombus must have sides that have the same "relationship" to vertices as one another. The easiest way to see this relationship is to consider lines of symmetry inherent in a parallelograms image. Image A has a vertical line of symmetry, but B has no line of symmetry (although, like all even ***n*** images, B does have 180° rotational symmetry). B is an example of even ***n*** and odd ***b*** image, the ONLY even/odd combination of ***n*** and ***b*** where there is no line of symmetry. The difficulty here can most easily be seen in the pattern of smallest line segments around the ***n***-gon which is a pattern of alternating even and odd (in B, the pattern is 2-1-2-1 clockwise around the 8-gon). The only difference in the no VT version of even ***n*** and odd ***b*** is that the pattern of smallest lines switches even and odd (so that in the alternative to B (with ***JKVW*** = 3,6,0,6), the pattern is 1-2-1-2 around the 8-gon and the resulting image is a mirror reflection of B). If the pattern is continuously alternating, there cannot be a line of symmetry. **To conclude: There is no line of symmetry when the image has even *n* and odd *b*.**

Claim: A line of symmetry exists whenever two successive smallest segments are the same size. That line of symmetry is midway between those segments.

Images C-E show the other possible even/odd combinations, each of which produces one or more rhombi. Each image has label at bottom left and two numbers at bottom right, the total number of rhombi as well as the number that are centrally located. Finally, each shows lines of symmetry in **red**. All three have ***b*** = 4, C has ***n*** = 9, while D and E have ***n*** =10.







Like A, image C has only one line of symmetry because ***n*** is odd, but these images show that it does not matter whether ***b*** is even or odd if ***n*** is odd, there is always a single line of symmetry given odd ***n***. With even ***n***, ***b*** must be even as well and either all four smallest ends are single line, like D, or all four smallest lines span two vertices, like E. In each instance, there are two lines of symmetry, midway between these smallest lines.

Finding Rhombi in Parallelograms Models. A rhombus requires adjacent sides that are the same size and lines of symmetry ensure that lines and angles on one side of the line are mirrored on the other side. Individual parallelograms from a parallelograms image that have opposing vertices on a line of symmetry must be a rhombus because adjacent line segments from the parallelogram whose corner is on the line of symmetry must be the same size as one another.

Just like squares in rectangles images, rhombi in parallelograms images require corners that are on a line of symmetry. And, just like centrally located squares, if all four corners are on lines of symmetry, then the rhombus is centrally located. As a result, no odd ***n*** images have central rhombi because they lack the second line of symmetry.

Out, In, and Mixed corners. Consider the outer corners of a line of symmetry. The outer corners can be described in one of three ways: *Out* means both outer corners are vertices like the ***b*** angles at vertices 0 and 5 in image D and ***c*** angles at 3 and 8 in E; *In* means both outer corners are not vertices of the ***n***-gon like the ***c*** angle vertices at 2.5 and 7.5 in D and the ***b*** angles at 0.5 and 5.5 in E; and *Mixed* means one corner is outer and one is inner like the ***b*** angles at 0 and 3.5 in A and the ***c*** angles at 2 and 6.5 in C.

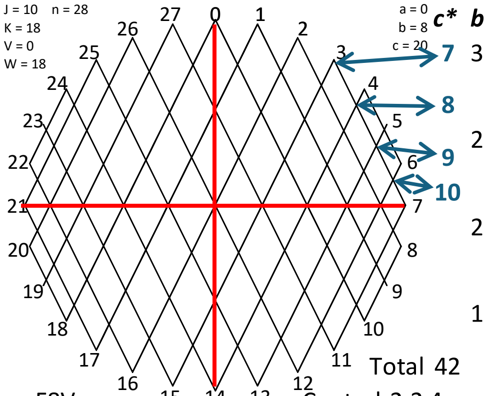
What Happens as *n* Changes? The three types of outer corners follow well-defined rules. Mixed corners only occur for odd ***n***. When ***b*** is also odd, then the line of symmetry is vertical like in A, but if ***b*** is even, the line of symmetry is from side to side. In the first instance, the outer corner is the top (vertex 0) and the inner corner is ***n***/2. In the second instance, the line of symmetry is from side to side connecting ***c*** angled corners. In particular, if ***n*** is odd then it is either of the form ***n*** = 4***k***+1 or 4***k***+3. The right side of the line of symmetry is at (***n***-1)/4 which is an outer ***c*** angled corner at ***k*** if ***n*** = 4***k***+1 and an inner ***c*** angled corner at ***k***+½ if ***n*** = 4***k***+3. In both instances the left side is the reverse (since it is ***n***/2 more and ***n*** is odd).

For even ***n***, if ***b*** is odd the two versions (VT and no VT) are simply mirror images of one another, but as we noted above, there are no rhombi. When ***b*** is even, ***b*** = 2***k***, and the two versions are distinct. The VT version has horizontal and vertical lines of symmetry, and the no VT version's lines of symmetry are tilted by a half vertex from vertical and horizontal. The smallest ***n*** is ***n*** = 2***b*** = 4***k*** in which case the rhombi are actually squares. The VT version of the image has ***k***×***k*** largest square which connects vertices 0-***k***-2***k***-3***k***-0 and the no VT version has a ***k***-1×***k***-1 largest square tilted by ½ vertex. As ***n*** increases from 4***k*** for fixed ***b***, the squares turn into largest central rhombi but additional non-central rhombi occur and it is worth noting that the largest non-central rhombi in the no VT version are ***k***×***k*** despite the largest central rhombus being ***k***-1×***k***-1.

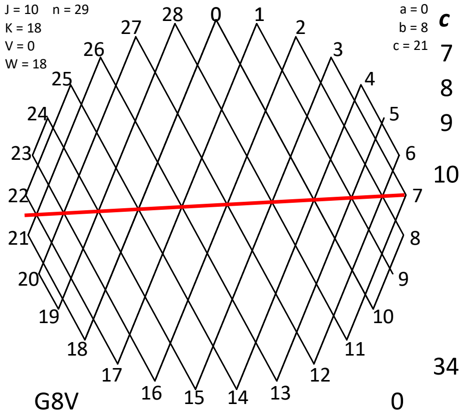
The in/out pattern as ***n*** increases from 4***k*** for ***b*** = 2***k*** remains fixed along the ***b*** line of symmetry. This line of symmetry only exists for even ***n***. There are ***b*** unit-sized rhombi along the ***b*** direction line of symmetry in the VT version with outer corners out, but ***b***-1 in the no VT version with outer corners in.

The in/out pattern as ***n*** increases from 4***k*** for ***b*** = 2***k*** cycles in the ***c*** direction line of symmetry every four ***n***. The pattern cycles from *out* to *mixed* to *in* to *mixed* for the VT version and *in* to *mixed* to *out* to *mixed* for the no VT version ***n*** = ***j*** mod 4 for ***j*** = 0, 1, 2, and 3. For both versions, the same pattern of increase occurs with respect to rhombus counts. The counts of rhombi with ***c*** angled corners on the ***c*** line of symmetry are the difference in triangular numbers. These values are same for 3 successive ***n*** values starting with out, followed by mixed and in. The next mixed (***n*** = 3 mod 4 for VT and ***n*** = 1 for no VT) is one larger and the next ***n***, out, is two larger than the previous plateau of 3 values. You can see this pattern in the 3 column by 4 row set of images.

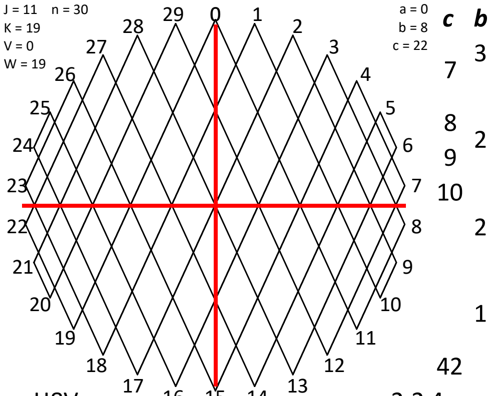
Individual cells have labels with 3 parts: Columns are F, G, H for ***n*** = 0, 1, and 2 mod 4 values of 28, 29, and 30; Rows 1 and 2 have ***b*** = 8 while 3 and 4 have ***b*** = 10; Rows 1 and 3 are VT and 2 and 4 are no VT. Each image has two (even ***n***) or one (odd ***n***) column of numbers to the right that are explained at the bottom. The top left image, F8V has four **blue arrows** explaining what the ***c*** column numbers refer to. There are **10** 1×1 rhombi, **9** 2×2 rhombi, **8** 3×3 rhombi, and **7** 4×4 rhombi, each with ***c*** angled corners on the horizontal line of symmetry. Since this image is out (***n*** = 0 mod 4 and the image is VT), the next two images have the same ***c*** counts. The ***b*** column counts for even ***n*** images F8V and H8V would be 4, 3, 2, 1 but the top two counts must be reduced by 1 (to 3, 2) because the ones with ***c*** angled corners on the horizontal line of symmetry, i.e., the central rhombi, have already been counted (as part of the numbers **7** and **9**). All three top row images have 34 ***c*** counted rhombi and both even ***n*** images have an additional 8 ***b*** counted rhombi or 42 total rhombi.



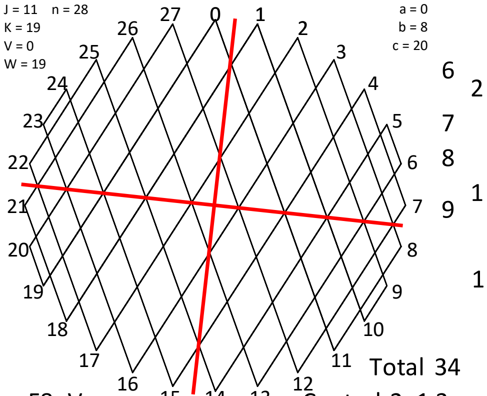
F8V Central 2:2,4



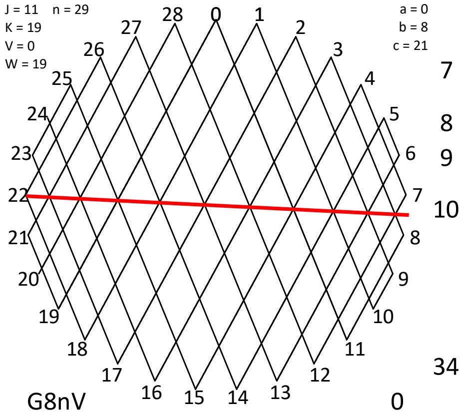
G8V 0



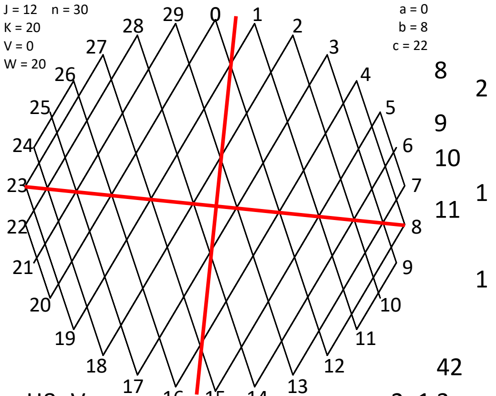
H8V 2:2,4



F8nV Central 2: 1,3



G8nV 0



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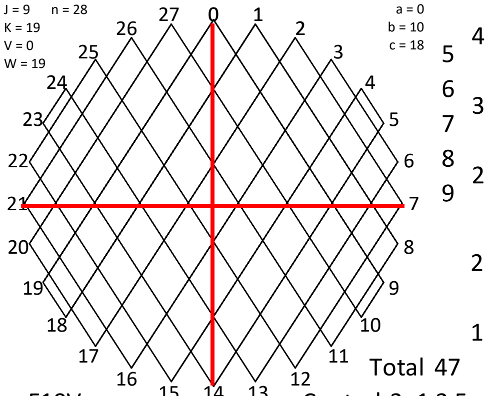
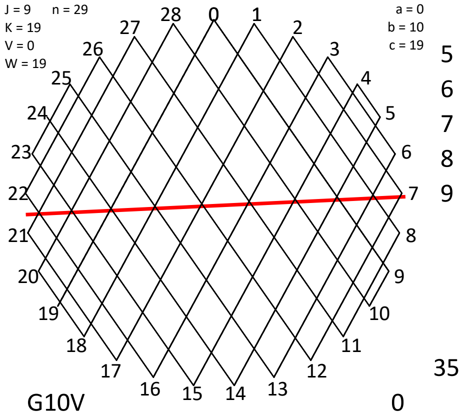
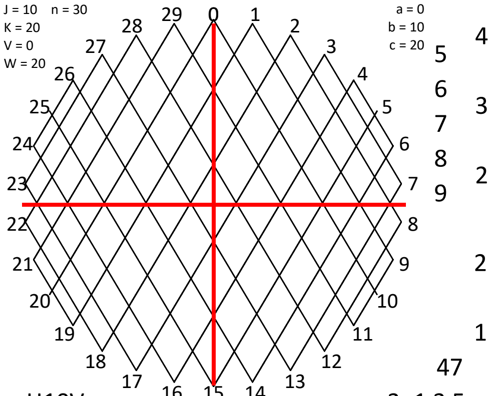


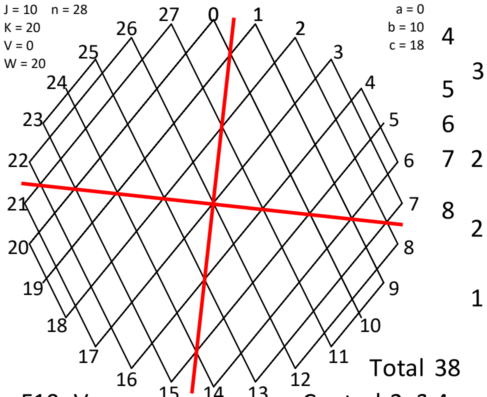
F10V Central 3: 1,3,5



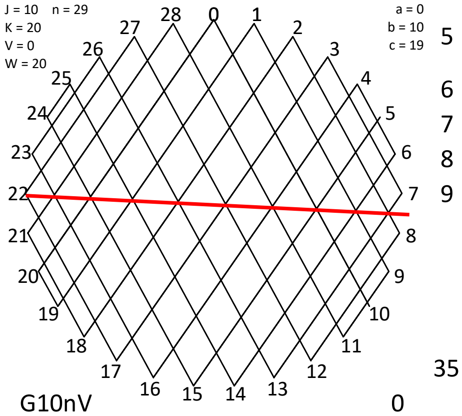
G10V 0



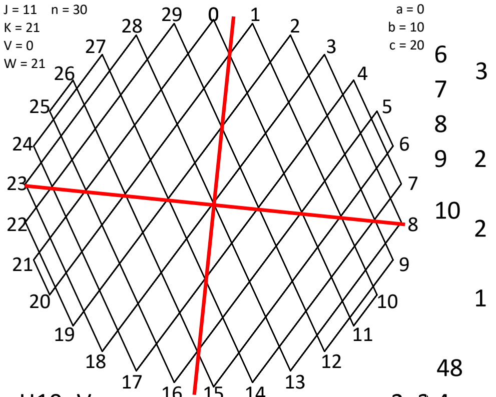
H10V 3: 1,3,5



F10nV Central 2: 2,4



G10nV 0



H10nV 2: 2,4

* ***c*** are rhombus apex counts by row in arcs with ***c*** angles on the line of symmetry. ***b*** are rhombus apex counts on the ***b*** line of symmetry not counted in the ***c*** column. Total is total rhombus count and Central is number and size that are centrally located.