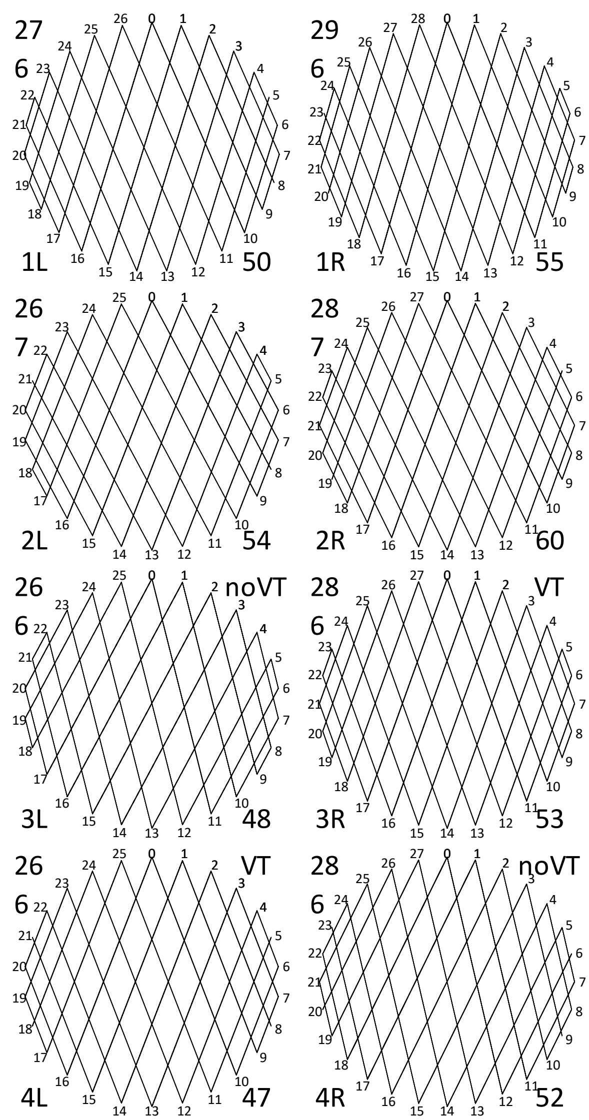
Counting Non-Overlapping Parallelograms in an Image

If the smaller angle in a parallelograms image is $\boldsymbol{b}$, then $\boldsymbol{n} \geq 2\boldsymbol{b}$. The easiest way to count parallelograms is to look at the parallel rows of parallelograms noting how the number of parallelograms in a row increases until it achieves the plateau value of $\boldsymbol{b}$-1. Four patterns are possible, depending on $\boldsymbol{n}$ and $\boldsymbol{b}$, and these patterns are examined in the 8 images below shown in a 4x2 array. Each image shows $\boldsymbol{n}$ and $\boldsymbol{b}$ in the upper left, a label in the lower left and the total non-overlapping parallelograms count in the lower right (which varies from 47 to 60). For each row, the right image has two larger $\boldsymbol{n}$ than the left and all rows have $\boldsymbol{b}$ = 6 except row 2 so that for rows 1, 3, and 4, the right count is 5 = $\boldsymbol{b}$-1 larger than the left.

Odd $\boldsymbol{n}$ = 2$\boldsymbol{k}$+1, images 1L and 1R. When $\boldsymbol{n}$ is odd there are ($\boldsymbol{n}$-1)/2 = $\boldsymbol{k}$ lines creating $\boldsymbol{k}$-1 rows of parallelograms in each direction. In both directions, the same pattern exists whether $\boldsymbol{b}$ is even or odd: the buildup to the plateau value occurs in $\boldsymbol{b}$-1 rows, one row each from 1 to $\boldsymbol{b}$-1. Either way, odd counts are on one side, and even counts are on the other. (For example, in 1L, odds start at 21-22 or 18-19 and evens start at 7-9 or 4-6 depending on direction chosen). The sum of these counts is $\Delta_{\boldsymbol{b}\text{-}1} = \boldsymbol{b}(\boldsymbol{b}\text{-}1)/2$. The rest of the rows, $\boldsymbol{k}$-1-($\boldsymbol{b}$-1) = $\boldsymbol{k}$-$\boldsymbol{b}$, have $\boldsymbol{b}$-1 parallelograms per row. Non-overlapping parallelograms counts are therefore $50 = \Delta_5+(6\text{-}1)(13\text{-}6)$ for 1L and $55 = \Delta_5+(6\text{-}1)(14\text{-}6)$ for 1R.

Even $\boldsymbol{n}$ = 2$\boldsymbol{k}$, odd $\boldsymbol{b}$, images 2L and 2R. Even $\boldsymbol{n}$ images have the same pattern on either end in a given direction, either both even or both odd. When $\boldsymbol{n}$ is even and $\boldsymbol{b}$ is odd, one direction has evens, the other has odds. The even direction has $\boldsymbol{k}$-2 rows, and the odd has $\boldsymbol{k}$-1 rows. Either direction can be used for counting and both will produce the same result, but odds are particularly easy because successive odds means perfect squares because of gnomons, discussed here. Notice that the odd direction changes as $\boldsymbol{n}$ changes by 2. Since $\boldsymbol{b}$ is odd, the plateau, $\boldsymbol{b}$-1, is even, and the ramp up has ($\boldsymbol{b}$-1)/2 rows on each side or a total of $2((\boldsymbol{b}\text{-}1)/2)^2 = (\boldsymbol{b}\text{-}1)^2/2$ (or 18 given $\boldsymbol{b}$ = 7). This leaves $\boldsymbol{k}$-$\boldsymbol{b}$ rows with $\boldsymbol{b}$-1 parallelograms per row, just like above. Given this, counts are $54 = 6^2/2+(7\text{-}1)(13\text{-}7)$ for 2L counting downward sloping rows and $60 = 6^2/2+(7\text{-}1)(14\text{-}7)$ for 2R counting upward sloping rows.



Even $\boldsymbol{n}$ = 2$\boldsymbol{k}$, even $\boldsymbol{b}$, single parallelogram ends, images 3L and 3R. The only differences with the even $\boldsymbol{n}$ and odd $\boldsymbol{b}$ are that now there are single parallelogram ends in both directions and since $\boldsymbol{b}$ is even, $\boldsymbol{b}$-1 is odd so the last step on the ramp up is the same height as the plateau, rather than a single parallelogram smaller. One can count this last step up as part of the ramp up OR as part of the plateau. Both answers are the same but the easiest to consider is a $\boldsymbol{b}$/2 ramp on each sides so total count from these two ramps is $2(\boldsymbol{b}/2)^2 = \boldsymbol{b}^2/2$ with $\boldsymbol{k}$-1-$\boldsymbol{b}$ remaining rows of $\boldsymbol{b}$-1 parallelograms or a total of $48 = 6^2/2+(6\text{-}1)(13\text{-}1\text{-}6)$ for 3L and $53 = 6^2/2+(6\text{-}1)(14\text{-}1\text{-}6)$ for 3R.

Even $\boldsymbol{n}$ = 2$\boldsymbol{k}$, even $\boldsymbol{b}$, double parallelogram ends, images 4L and 4R. As noted above, even directions have $\boldsymbol{k}$-2 rows. Since $\boldsymbol{b}$ is even, the plateau, $\boldsymbol{b}$-1 is odd, so like in row 2, there is no ambiguity as to whether the last row on each ramp should be part of the ramp or the plateau. There are $\boldsymbol{b}$/2-1 even rows on each side of successive even numbers or $2\Delta_{\boldsymbol{b}/2\text{-}1}$ per side. Both sides sum to $2(\boldsymbol{b}/2\text{-}1)\boldsymbol{b}/2 = \boldsymbol{b}^2/2\text{-}\boldsymbol{b}$. The $\boldsymbol{k}$-2-2($\boldsymbol{b}$/2-1) = $\boldsymbol{k}$-$\boldsymbol{b}$ rest of the rows have $\boldsymbol{b}$-1 parallelograms per row, or a total of $47 = 6^2/2\text{-}6+(6\text{-}1)(13\text{-}6)$ for 4L and $52 = 6^2/2\text{-}6+(6\text{-}1)(14\text{-}6)$ for 4R.

The upper right of corner row 3 and 4 images notes that VT and no VT images must alternate to maintain smallest lines by row.

The intuition behind row 4 having one less parallelogram than 3. Row 4 images have 4 vertices that are not part of a parallelogram. By contrast, all vertices in row 3 images are part of parallelograms.