

Counting Rhombi in Parallelograms Images

We know that parallelograms images have two angles, b and c which sum to n . We saw in the last section that even n images require even b (and hence even c) to have rhombi. We focus on the VT version of even b images here, with reference guides being rows 1 and 3 of images F-H from the previous section. Rather than repeat these images, we simply refer to them here. The even b no VT and the odd n , odd b analyses are left as challenge questions.

The same equations work for $n = 4j$ and $n = 4j+2$ where $j \geq k = b/2$, the only difference is the first is *out* on the horizontal diameter (connecting vertices j and $3j$) and the second is *in* on the horizontal diameter (connecting $j+0.5$ and $3j+1.5$). The value of j is the integer portion of $n/4$, $j = \text{INT}(n/4)$.

Define the maximum value $M = (4j-2k)/2 = 2j-k$. This is the number of 1×1 rhombi on the horizontal diameter. This number declines by 1 each as 2×2 rhombi or 3×3 rhombi, etc. are counted on this diameter. There are k such arcs of apex counts (the top arc of apexes are vertices). This is what is being counted in the c columns to the right of images F, G, and H for $n = 28, 29$, and 30 . The total number of such peaks is therefore given by the difference between two triangular numbers:

$$c \text{ column: Total rhombi with } c \text{ corners on the } c \text{ diameter} = \Delta_M - \Delta_{M-k} = M(M+1)/2 - (M-k)(M-k+1)/2 = k(4j-3k+1)/2.$$

To this, one must add the number of rhombi with peaks (b corners) on the vertical diameter whose c corners are not on the horizontal diameter. If you count from the bottom up, you quickly see that the progression is 1, 2, ... until the peak is above the horizontal diameter, then the progression is reduced by 1 per peak to avoid double counting. This is what is being counted in the b columns to the right of images F and H for even $n = 28$ and 30 .

$$b \text{ column: Total rhombi with } c \text{ corners off the } c \text{ diameter} = \Delta_k - \text{INT}((k+1)/2) = k(k+1)/2 - \text{INT}((k+1)/2).$$

The total number of rhombi in an even b VT image is the sum of these two values when $n = 0$ or $2 \pmod 4$ (i.e., n is even); when $n = 1 \pmod 4$, the total count is the first value; and when $n = 3 \pmod 4$, the total is k larger than for $n = 1 \pmod 4$ since each of the k c diameter corner counts increase by 1.

When n and b are even, the number of rhombi that are centrally located is $\text{INT}((k+1)/2)$.

Columns F-H, Rows 1 and 3 Examples. All 12 images have $j = \text{INT}(n/4) = 7$ given $n = 28, 29$, and 30 .

The first row is $b = 8$ so $k = 4$. Images F8V, G8V, and H8V have c column counts of $4(4 \cdot 7 - 3 \cdot 4 + 1)/2 = 2 \cdot 17 = 34$.

F8V and H8V have additional b column counts of $4 \cdot 5/2 - \text{INT}((4+1)/2) = 10 - 2 = 8$.

So total counts in Row 1 are: 42, 34, and 42 for F8V, G8V, and H9V. For $n = 31$ (not shown), the count is $34+4 = 38$.

The third row is $b = 10$ so $k = 5$. Images F10V, G10V, and H10V have c column counts of $5(4 \cdot 7 - 3 \cdot 5 + 1)/2 = 5 \cdot 14/2 = 35$.

F10V and H10V have additional b column counts of $5 \cdot 6/2 - \text{INT}((5+1)/2) = 15 - 3 = 12$.

So total counts in Row 3 are: 47, 35, and 47 for F10V, G10V, and H10V. For $n = 31$ (not shown), the count is $35+5 = 40$.

Columns F and H have $k = 4$ and 5 so $\text{INT}((k+1)/2) = 2$ in row 1 and 3 in row 3 meaning there are 2 central rhombi for F8V and H8V but 3 for F10V and H10V.

A Rotated Example. The image to the right is not in the standard format but it is easy to see how many rhombi are in the image. Even without the added lines, it is apparent that $b = 6$ (so $k = 3$) and therefore there are rhombi in the image. The **red** diameter line of symmetry connecting c angles has 11 1×1 rhombi (five on each side of the other **red** diameter line of symmetry passing through b angles); since the b corners are *out* the image is VT; and since the outer c corners are *out*, $n = 0 \pmod 4$ (by counting $1/4$ of the vertices, we see that $n = 28$, so $j = 7$). The **blue** lines show the outer edges of rhombi that result; the 11, 10, and 9 shown at right are the c column values for this image. The b values are $\Delta_3 - 2 = 4$ so the total number of rhombi in this image is:

$$\text{Total count} = 11+10+9+4 = 34, \text{ 2 of which are } 1 \times 1 \text{ and } 3 \times 3 \text{ central rhombi.}$$

