MA. Counting Parallelogram: Triangles of Triangular Numbers

Counting parallelograms is not as easy as counting triangles but with practice it can be mastered. We will not cover all cases here but will provide sufficient breadcrumbs to see the patterns that develop. These patterns lean heavily on triangular numbers first discussed here, and arcs of apexes which appear throughout this analysis.

Since we are counting parallelograms using the upper smaller angle of the parallelogram as its distinguished point, apexes counts use vertices along the top between the two smallest lines (at vertices 3 and 17 in Figure 1) the other apexes are the top ***b***-2 arcs, since the bottom apex arc forms the larger angles of the bottom set of parallelograms (like the arc starting just above vertices 7 and 13 in Figure 1.

Three Colored Counts. Figure 1 uses the standard setup for creating parallelogram images given ***n*** = 19, ***b*** = 6 and no VT = 1. The parallelogram counts are noted in three colors, purple, green and red rather than a single color.

Before considering the colors, quickly scan the counts arc by arc. From bottom to top, each arc is bounded by a triangular number, noted in the second column of Table 1 by t_s. Given ***b*** = 6, the largest count in any cell is t_5 = 15.

Purple. The triangle of numbers bounded by vertices 7, 0 and 13 is a "5-high" triangle of triangular numbers, denoted T5. There are 5 t_1s, 4 t_2s 3 t_3s, 2 t_4s and 1 t_5. These sum to T5 = 70.

Green. The line of numbers whose apexes are on the line from vertex 1 to 6 is a "5-high" line of triangular numbers, denoted L5. The sum of these numbers is L5 = 35.

Red. The rest of the parallelogram apex counts are neither Ts or Ls but they are related to these counts as we will soon see. For Figure 1, that sum is 70 so that the total number of parallelograms in the image is 175.

1. Defining Triangles and Lines of triangular numbers

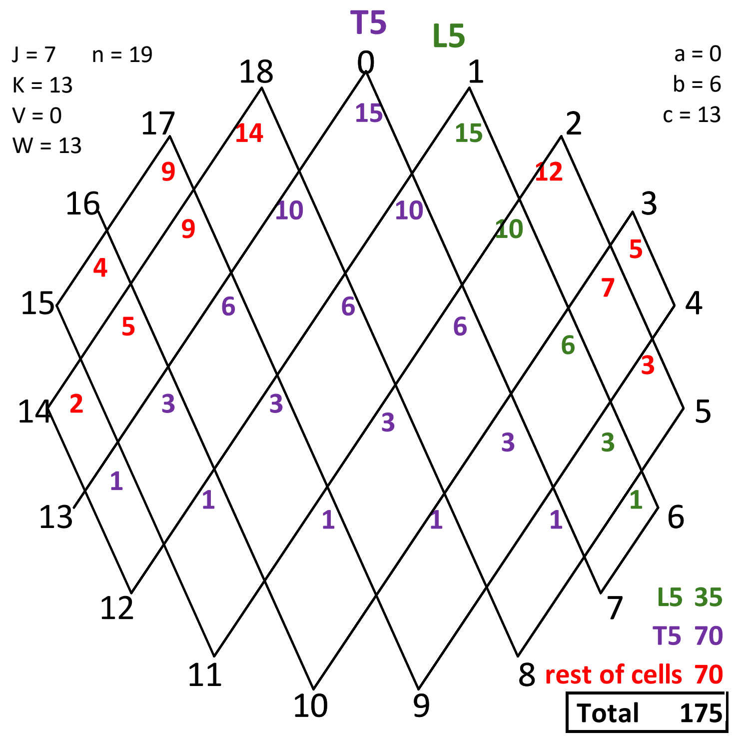


Table 1. The first 10 values of t, L and T are shown in columns 2 through 4 of Table 1. Each of these columns is created in exactly the same way using a *sideways sum* where the value in any cell below row 1 is the value from the cell above plus the number from the cell just to the left. For example, the fifth triangular number, $t_5 = t_4 + 5 = 15$. Similarly, the sum of 5 triangular numbers, L5 = L4 + t_5 = 35 and the fifth triangle of triangular numbers, T5 = T4 + L5 = 70.

Size of Δ				1. Triangles and Lines of Triangular Numbers										Ls = s(s+1)(s+2)/6		Sloane
s	Δs = t_s	line Σ = Ls	Σ of Σ = Ts	1	2	3	4	5	6	7	8	9	10	Δs = t_s	C&E Eq.21	A000332
1	1	**1**	**1**	1	2	3	4	5	6	7	8	9	10	1	**1**	**1**
2	3	**4**	**5**		3	6	9	12	15	18	21	24	27	3	**4**	**5**
3	6	**10**	**15**			6	12	18	24	30	36	42	48	6	**10**	**15**
4	10	**20**	**35**				10	20	30	40	50	60	70	10	**20**	**35**
5	15	**35**	**70**					15	30	45	60	75	90	15	**35**	**70**
6	21	**56**	**126**						21	42	63	84	105	21	**56**	**126**
7	28	**84**	**210**							28	56	84	112	28	**84**	**210**
8	36	**120**	**330**								36	72	108	36	**120**	**330**
9	45	**165**	**495**									45	90	45	**165**	**495**
10	55	**220**	**715**										55	55	**220**	**715**
Multiple ways to get triangles and lines of trianglular numbers				**1**	**5**	**15**	**35**	**70**	**126**	**210**	**330**	**495**	**715**	**C(s+1,2)**	**C(s+2,3)**	**C(s+3,4)**
				Eq in E3: =IF(E$2>=$A3,(E$2+1-$A3)*$B3,"")										**Ts = s*(s+1)*(s+2)*(s+3)/24**		

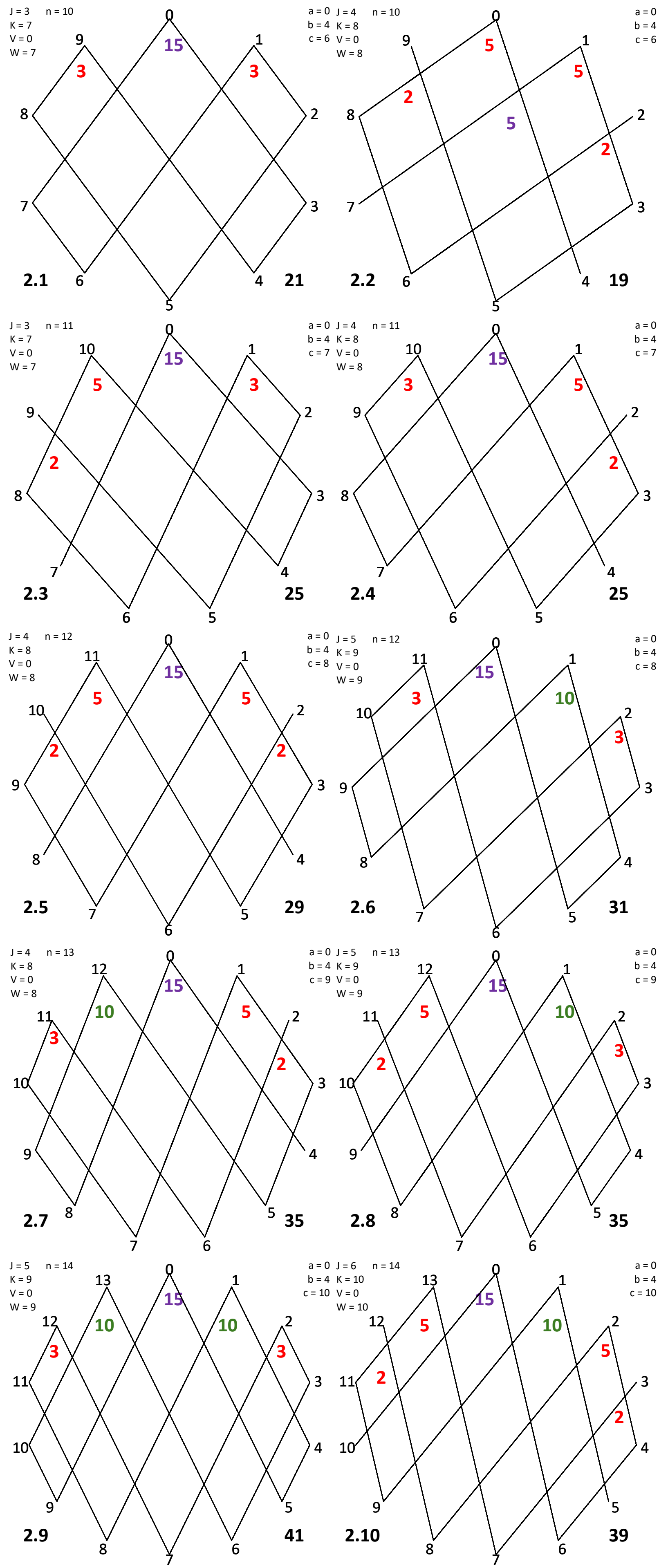
The next 10 columns show a second way to calculate **T**, and the final three columns show that t_s **Ls**, and **Ts** can be calculated using binomial combinations. [For a geometric proof that the sum of s triangular numbers, **Ls** = s(s+1)(s+2)/6, see the discussion leading to equation 21 from Chakerian and Erfle, *Up the Hill and Down Again*, 2023.] It is worth noting that **t**, **L**, and **T** are the 3rd, 4th and 5th diagonals of *Pascal's Triangle*.

Smallest Lines Patterns. Figure 2 shows 10 images in a 2x5 array with the left column being the VT version and the right being the no VT version using the standard parallelograms setup for ***b*** = 4 with ***n*** ranging from 10 to 14. The starting value ***n*** = 10 was chosen because it is the smallest ***n*** for which there is a ***b***-1 triangle of triangular numbers at 0. This occurs when ***n*** = 3***b***-2 in the VT style images, see 2.1, and ***n*** = 3***b***-1 for no VT images like 2.4.

The pattern of smallest lines across images cycles every four ***n***. Given VT and no VT switching even and odd, it is unsurprising that the patterns by row exactly oppose one another. From the first time there is a triangle of triangular numbers at vertex 0 the pattern *clockwise* around the vertices is: 1,1,1,1 to 1,1,2,2, to 2,2,2,2 to 2,2,1,1 for VT images with 2.9 having the same smallest lines pattern as 2.1 with 20 more parallelograms due to 2 additional **L3** = **10** at vertices 1 and 13. The no VT smallest lines pattern starts at 2.4 going from 2,2,1,1 to 1,1,1,1 to 1,1,2,2, to 2,2,2,2.

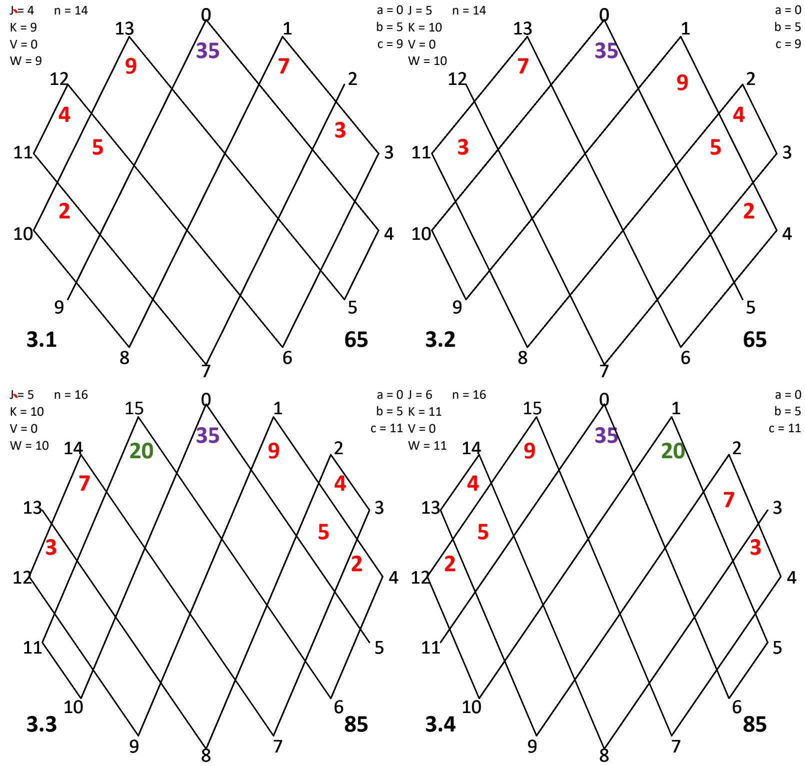
Even versus Odd *n* counts. If you compare the odd ***n*** images in the second and fourth rows of Figure 2 you will quickly note that they are mirror images of one another and hence have the same total parallelograms count.

The same is not true for the even ***n*** rows 1,3 and 5. The reason for this difference keys off of the same issue uncovered in Section 6.11. Even ***b*** and even ***n*** images **B** and **C** there have a different number of interior apexes in the VT and no VT style, just like here. And notice that all three even rows have the same sized smallest lines along the top (for example, both 1 for 2.1 but both 2 for 2.5). The larger of the two counts switches back and forth between VT and no VT with Figure 2.1 being 2 larger than 2.2 but 2.6 is 2 larger than 2.5. Note that the larger of the two counts occurs when there are no open ends of lines (for example, there are



single line vertices at 2, 4, 7, and 9 in 2.2 but none in 2.1). These open ended lines mean that there is one fewer *smallest* parallelogram in 2.2 than 2.1 (there are 8 in 2.1 and 7 in 2.2).

Even ***n*** values can also have the same total parallelograms count as we see in Figure 3. When an even ***n*** is based on odd ***b*** like Figure 3 which shows ***b*** = 5 for ***n*** of 14 and 16, the counts are the same. The smallest lines patterns starting at ***n*** = 13 for VT is 2,1,1,2 to 2,1,2,1 to 1,2,2,1 to 1,2,1,2 of which the 2nd and 4th are 3.1 and 3.3 and starting at ***n*** = 14 for no VT is 1,2,1,2 to 2,1,1,2 to 2,1,2,1 to 1,2,2,1 of which the 1st and 3rd are 3.2 and 3.4. (Note that in terms of smallest lines for both even or odd ***b***, if ***n*** is even, the 1st and 3rd values are the same and the 2nd and 4th are the same but if ***n*** is odd, 1st and 3rd are different, and the 2nd and 4th are different.)



In this instance, the VT and no VT images are mirrors of one another, much like the odd ***n*** images in Figure 2. In both instances, the top smallest lines are not the same with one being 2 and the other being 1. These smallest lines determine whether the tilted columns have an even or an odd number of entries. This is the topic to which we now turn.

A Deeper Dive into the Red Cells. The red cells in Figures 1-3 are organized into columns that are parallel to the outside lines of **T** and **L**. The easiest pattern to see are the ones with bottom at the upper part of a 2 line segment. The upper edge of these columns are lines from vertices 18 to 14 in Figure 1; 10 to 8 in 2.3; 1 to 3 in 2.4; 1 to 3 and 11 to 9 in 2.5; 1 to 3 in 2.7; 12 to 10 in 2.8; 2 to 4 and 13 to 11 in 2.10; **13 to 10 in 3.1**; 1 to 4 in 3.2; 1 to 4 in 3.3; and 15 to 12 in 3.4. In each case, there are ***b***-2 parallelograms in the column, and each entry is 1 smaller than a triangular number because each is missing a parallelogram that is partially visible with peak at the bottom of the vertex mentioned.

Focus on Figure 3.1. If you consider vertex 10 in 3.1 to be the peak of a parallelogram the right wide angle would be at the intersection of lines 10 to 8 and 0 to 9. There is NOT a parallelogram in the image (since its other vertices extend outside the polygon) but if you imagine one at that location, then each apex in the 13 to 10 column would be a triangular number or that column would be **L4**. The absence of this parallelogram creates a column with upper left sides along the line from vertices 13 to 10 having values one less than triangular numbers or 9, 5, and 2 which can be reimagined as:

9+**5**+**2** = 16 = (10-1)+(6-1)+(3-1) = $(t_4-t_1)+(t_3-t_1)+(t_2-t_1) = t_4+t_3+t_2 - 3t_1 + (t_1-t_1)$ = **L4**-$3t_1$-**L1** since t_1 = **L1**.

Similarly, the numbers in the columns 1 to 3 and 12 to 11 can be reimagined as:

7+**3** = 10 = (10-3)+(6-3) = $(t_4-t_2)+(t_3-t_2)) = t_4+t_3 - 2t_2$ + (L2-L2) = **L4**-$2t_2$-**L2,** and

4 = 10-6 = 10-t_3 + (**L3**-**L3**) = **L4**-t_3-**L3**.

Adding the three columns together we obtain: Sum of **reds** = **16**+**10**+**4** = **30** = 3**L4** – $(3t_1+2t_2+t_3)$ – (**L1**+**L2**+**L3**) = 3**L4**-2**T3**.

This same strategy works for other values of ***b*** so long as the red areas are not symmetric, or to put it another way, if one has smallest size 1 and the other has smallest size 2. In terms of Figure 2, this works for the second and 4th rows as well as Figure 1 and all images in Figure 3.

More generally, if an image has smallest angle ***b***, with a ***b***-1 triangle of triangular numbers at vertex 0, **T*b*-1**, and zero or more ***b***-1 long lines of triangular numbers, **L*b*-1**, surrounding vertex 0, with top side smallest-sized lines that are not the same length, one can add up **T*b*-1**, **L*b*-1**s and the **red areas** having a number of parallelograms totaling (***b***-2)**L*b*-1** – 2**T*b*-2**.

Finally, note that one can further simplify this result because T*b*-1 = T*b*-2 + L*b*-1 so one need only subtract T*b*-2 from the right number of L*b*-1s to find the final answer. They were kept separate in the above discussion because the T*b*-1 comes from vertex 0 and the T*b*-2s come from the **red areas** outside the **T** and **L**s at the top of the image.

Figure 4 has 12 images in a 4x3 array with 4 columns: ***b*** = 8, 9, 10 and 11; and 3 rows, ***n*** = 29, 30, and 31. Five of the images, 4.1, 4.3, 4.6, 4.9, and 4.11 satisfy the requirements laid out in the previous paragraph and have total parallelogram counts ranging from 714 to 1320 noted in bottom right with calculations noted in bottom left of each image (except 4.1). (The rest show between outer **red / ** lines counts.) The calculation for 4.1 is as follows:

Total(***n*** = 29, ***b*** = 8) = **T7** + **3L7** + (**6L7-2T6**) = **210** + (**3·84**) + (**6·84-2·126**) = 714.

A General Rule given *n* > 3*b* and *bn* is Odd. When n_0 = 3***b***-1, if ***b*** is odd then n_0 is even and if ***b*** is even then n_0 is odd. (Examples of n_0 images are 2.3 and 2.4, 3.1 and 3.2, and 4.3.) In each instance, the image has a **T*b*-1** at vertex 0 with no **L*b*-1**s and the smallest lines in the upper right and upper left differ from one another so that there are ***b***-2 lines of parallelograms in the red areas above the triangle of triangular numbers, one each of length 1 to ***b***-2. The total number of parallelograms in this instance is:

Total(n_0 = 3***b***-1) = **T*b*-1** + (***b***-2)**L*b*-1** – 2**T*b*-2** = (***b***-1)**L*b*-1** – **T*b*-2**.

As ***n*** increases by 2 from there, two vertices are added to the image and a ***b***-1 long line of triangular numbers, **L*b*-1**, is added as well. Therefore, for a general ***n*** that is an even number of vertices more than n_0, we can write this as ***n*** = n_0+2***k***, and the total number of parallelograms is:

Total(***n*** = 3***b***-1+2***k***) = (***b***-1+***k***)**L*b*-1** – **T*b*-2**.

This can be calculated using the values in Table 1 or rewritten as a function of ***b*** and ***k*** using the equations for **L** and **T** noted in Table 1 as:

Total(***b***, ***k***) = (***b***-1+***k***)·(***b***-1)·***b***·(***b***+1)/6-(***b***-2)·(***b***-1)·***b***·(***b***+1)/24

Three examples that key off Figure 4 images. If the middle row was extended one to the left of image 4.5 you would have ***n*** = 30 and ***b*** = 7. Given ***b*** = 7, n_0 = 20 so ***k*** = 5 and the number of parallelograms is:

Total(***n*** = 30, ***b*** = 7) = (7-1+5)**L6** – **T5** = 11·56-70 = 546.

Similarly, two to the left of image 4.9 would be ***n*** = 31 and ***b*** = 6. Given ***b*** = 6, n_0 = 17 so ***k*** = 7 and the number of parallelograms is:

Total(***n*** = 31, ***b*** = 6) = (6-1+7)**L5** – **T4** = 12·35-35 = 385.

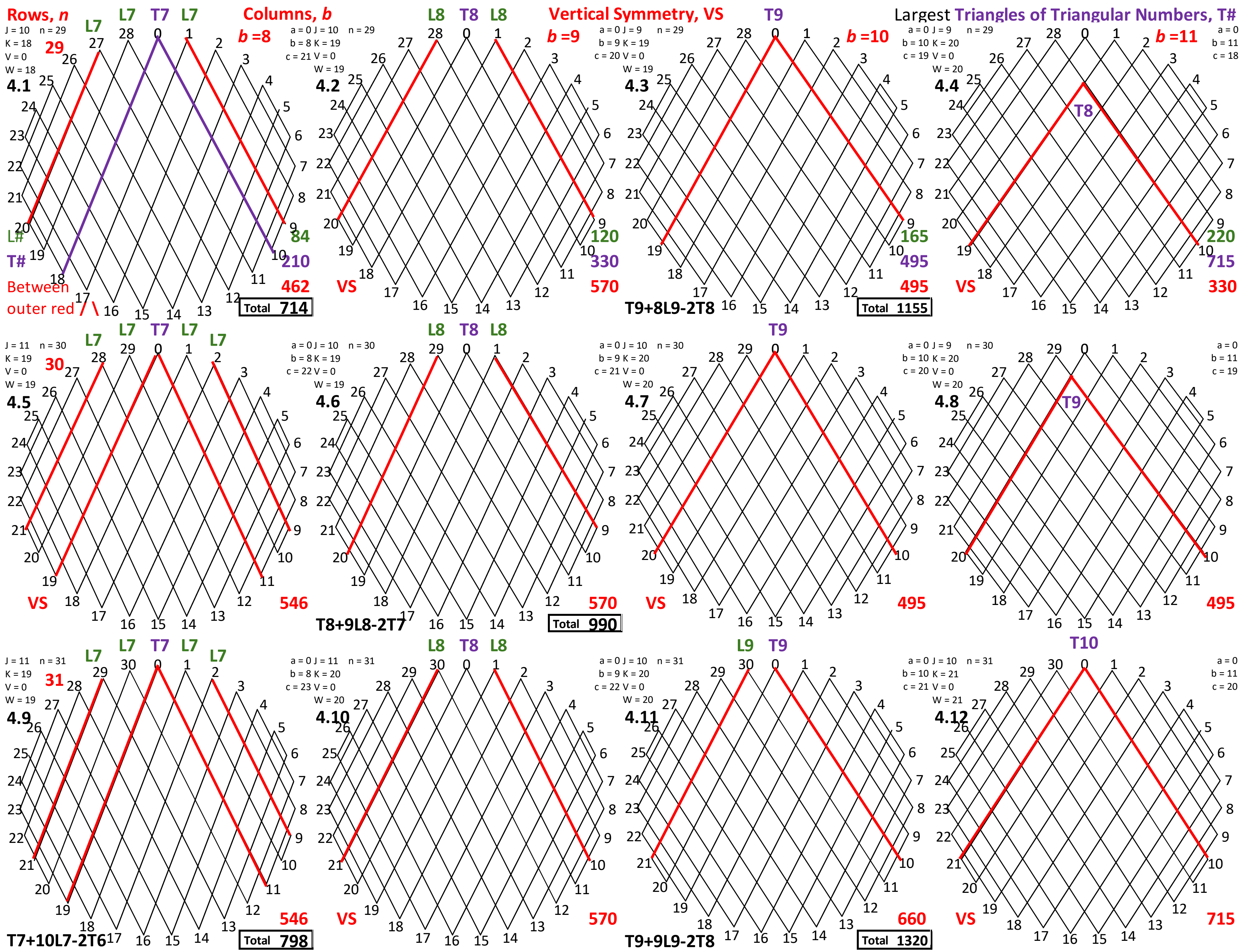
Finally, just below image 4.12 (which has smallest lines of two on upper right and left) is n_0 = 3***b***-1 = 32 given ***b*** = 11. In this instance, ***k*** = 0 so the number of parallelograms is:

Total(***n*** = 32, ***b*** = 11) = (11-1+0)**L10** – **T9** = 10·220-495 = 1705.

Beyond T*b*-1 with Even and Odd Smallest Lines. Images like 4.4 or 4.8 which have ***n*** < 3***b***-2 for VT style or 3***b***-1 for no VT style like image 2.2 do not have **T*b*-1** at vertex 0. Nonetheless, if ***bn*** is odd, you can calculate the areas above the lower degree **T** using variations on the methodology described above. To see how, consider image 4.8. Everything above the **T9** triangle can be visualized as 9 rows of parallelograms, one each from length 1 (with top right edge between vertices 4 to 5) to 9 (with top right edge from 0 to 9) with odd length lines using vertices 0 to 9 and even length lines using vertices 29 to 21. As a result, the total parallelograms in image 4.8 is the sum of **T9** and the numbers in the 9 lines of parallelograms above **T9** (which is 9**L10**-2**T9** for the reasons discussed above), or:

Total(***n*** = 30, ***b*** = 11) = **T9** + (9**L10**-2**T9**) = 9**L10**-**T9** = 9·220-495 = 1485.

Additionally, images with vertical symmetry (noted with **VS** in the VT Figure 4 images) or no VT images with symmetry between 0.5 and ***n***/2+0.5 (like images 2.2, 2.6 and 2.10) must be dealt with differently because in this instance, one only has all even or all odd length lines of parallelograms areas above the **red / ** lines since they do not alternate even and odd from side to side in the way we have discussed above. These calculations are left as an open exercise.



Rows, n
Columns, b
Vertical Symmetry, VS
Largest Triangles of Triangular Numbers, T#
4.1
J = 10 n = 29 K = 18 V = 0 W = 18
29
L7 L7 T7 L7
L#
T#
84
210
Between outer red / \
462
Total 714
4.2
b =8
a = 0 J = 10 n = 29 b = 8 K = 19 c = 21 V = 0 W = 19
L8 T8 L8
120
330
VS
570
4.3
b =9
a = 0 J = 9 n = 29 b = 9 K = 19 c = 20 V = 0 W = 19
T9
165
495
495
T9+8L9-2T8
Total 1155
4.4
b =10
a = 0 J = 9 n = 29 b = 10 K = 20 c = 19 V = 0 W = 20
b =11
a = 0 b = 11 c = 18
T8
220
715
VS
330
4.5
J = 11 n = 30 K = 19 V = 0 W = 19
30
L7 L7 T7 L7 L7
VS
546
4.6
a = 0 J = 10 n = 30 b = 8 K = 19 c = 22 V = 0 W = 19
L8 T8 L8
T8+9L8-2T7
570
Total 990
4.7
a = 0 J = 10 n = 30 b = 9 K = 20 c = 21 V = 0 W = 20
T9
VS
495
4.8
a = 0 J = 9 n = 30 b = 10 K = 20 c = 20 V = 0 W = 20
a = 0 b = 11 c = 19
T9
495
4.9
J = 11 n = 31 K = 19 V = 0 W = 19
31
L7 L7 T7 L7 L7
T7+10L7-2T6
546
Total 798
4.10
a = 0 J = 11 n = 31 b = 8 K = 20 c = 23 V = 0 W = 20
L8 T8 L8
VS
570
4.11
a = 0 J = 10 n = 31 b = 9 K = 20 c = 22 V = 0 W = 20
L9 T9
T9+9L9-2T8
660
Total 1320
4.12
a = 0 J = 10 n = 31 b = 10 K = 21 c = 21 V = 0 W = 21
a = 0 b = 11 c = 20
T10
VS
715